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Optimal Power Allocation in Wireless Communication Networks Using Stochastic Control

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Abstract: Wireless communication networks operate over time-varying channels that depend on channel gains, power allocation algorithms, and inter-channel interference. In this paper, We present techniques for computing the optimal control inputs, which lead to the attainment of the intended Signal to Interference plus Noise ratio (SINR) at different receivers. This optimization problem can be efficiently resolved using non-linear programming tools and permits the simultaneous assignment of multiple desired SINRs at different instances. We establish the conditions necessary to ensure the feasibility of the optimal control problem.

Keywords: Signal to Interference Noise Ratio, Optimal Control, Wireless Communication, Stochastic Control, Coupled Riccati Equations

1. INTRODUCTION

The development of effective power allocation algorithms is a vital component of wireless network planning. For instance, because wireless communication operates on a broadcast principle, signals can interfere with one another. Efficient power allocation, therefore, plays a critical role in optimizing spectral utilization and enhancing the end-user experience. Additionally, it aids in reducing overall energy consumption through precise transmission power management. Considering the inherent uncertainty and fluctuations in wireless channel conditions, maintaining an adequate minimum signal reception level is imperative for reliable communication, even in the absence of interference. Various algorithms have been created for power control purposes, aiming to reduce the number of channels needed to accommodate all users Papavassiliou and Tassiulas (1995), enhance the Signal-to-interference plus Noise (SINR) ratio while constraining the total transmitted power Grandhi et al. (1994); Goldsmith and Varaiya (1997), and minimize the overall power allocations Yates and Huang (1995); Rulnick and Bambos (1997). In Goodman and Mandayam (2000), the distributed power control problem is solved using a game-theoretic approach, where the optimal power allocation is defined by a Nash equilibrium. Stochastic-based online algorithms are proposed for controlling the transmission powers for time-varying channels Holliday et al. (2003, 2004). A Kalman filter approach for solving the optimal power allocation is derived in Leung (2002) by using the predicted interference levels and the estimated channel gains.

In this work, we address this power allocation challenge through a stochastic optimal control approach that considers the uncertainties in power allocations. The proposed solution applies to both fixed channels and time-varying channels.

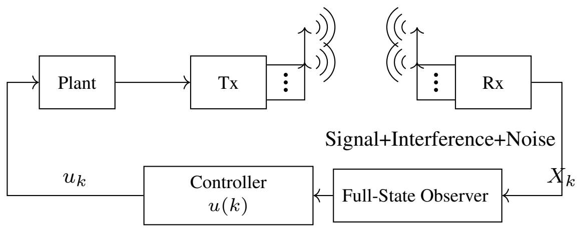


Figure 1. Network-controlled system with noisy channel

Figure 1 displays the block diagram of a network-controlled system, in which the plant controls the transmission powers using a stochastic-based time-varying allocation method Holliday et al. (2003, 2004). The received signals are monitored by the observer block, and for simplicity, we assume that the system is fully observable. In the absence of the controller, the plant determines power allocation based on the signals received from the observer and the channel matrix to attain the desired SINR. This approach is known as Foschini's distributed power control algorithm for SINR optimization, as detailed in Foschini and Miljanic (1995). However, if there is a need to modify the desired SINR, then redesigning the channel matrix is necessary to achieve the updated SINR requirements. To avoid these complexities, we utilize control inputs to adjust the channel matrix, ensuring that the desired SINR is attained.

2. PROBLEM FORMULATION

Consider a wireless communication system with n transmitters, $\{t_1, t_2, \cdots t_n\}$ and n receivers, $\{r_1, r_2, \cdots r_n\}$. In this network with n interfering links, the SINR for the i^{th} user for fixed channel gain is defined as Holliday et al. (2003)

$$R_i = \frac{G_{ii}S_i}{\eta_i + \sum_{i \neq j} G_{ij}S_j}, \quad (1)$$

where $G_{ij} > 0$ is the power gain from the transmitter of the j^{th} link to the receiver of the i^{th} link, S_i is the power of the i^{th} transmitter, and η_i is the noise power at the i^{th} receiver. When each link is assumed to have a minimum SIR requirement of $\gamma_i > 0$, then the constraint can be written in matrix form as

$$(I - F)S > w, \quad \text{with} S > 0, \quad (2)$$

where $S = (S_1, S_2, \cdots, S_n)^T$ is the column vector of transmitted powers and u is the column vector of noise powers, and F is a matrix with

$$F_{ij}(k) = \begin{cases} 0, & \text{if} \quad i = j, \\ \frac{\gamma_i G_{ij}(k)}{G_{ii}(k)}, & \text{if} \quad i \neq j. \end{cases}$$

The Foschini-Miljanic algorithmFoschini and Miljanic (1995) states that the following iterative power control algorithm in (3) converges to the optimal power allocation when the Perron-Frobenius eigenvalue of F is less than 1.

$$S(k+1) = FS(K) + w. \quad (3)$$

The drawback of Foschini's algorithm is that the channel gain matrix is fixed and the Perron-Frobenius eigenvalue of the channel gain matrix should be less than 1. Moreover, the noise input is assumed to be fixed for all time t, and the fluctuations of the transmitted power are not taken into account. In this study, we assume a continuous stochastic power allocation algorithm following Foschini's approach as

$$dx(t) = A(t)x(t)dt + Gdw(t), \quad (4)$$

where $x_i(t)$ denotes the transmitted power of the i^{th} transmitter at time t, the matrix $A(t) \in \mathbb{R}^{n \times n}$ denotes the time-varying channel gain matrix and $w(t) \in \mathbb{R}^p$ represents the standard $p-$dimensional Brownian motion with noise input matrix $G \in \mathbb{R}^{n \times p}$. We assume that the initial transmitted power fluctuates around its mean with an initial covariance matrix, $\Sigma(0) = \Sigma_0$. Further, we assume that

$$\mathbb{E}(w(t)) = 0; \;\; \mathbb{E}[w_{t_1} w'_{t_2}] = \delta_{t_1 t_2} I_p; \; \mathbb{E}[x_{t_1} w'_{t_2}] = 0,$$

where $\mathbb{E}$ is the expectation, $\delta_{t_1 t_2} = 1$, I_p is an identity matrix of order p and $t_1, t_2 \in t$. We use X' to denote the transpose of a vector or matrix X. Let ρ denote the joint probability distribution function (pdf) of the initial state $P(0)$, and ρ_1, $\rho_2, \cdots, \rho_n$, the marginal pdfs for the states $x_1, x_2, \cdots x_n$. The SINR for each receiver at time t, denoted as $\alpha_i(t)$ is defined as Holliday et al. (2003); Foschini and Miljanic (1995); Holliday et al. (2004)

$$\alpha_i(t) = \frac{A^t(i,i)\sigma_{ii}(t)}{\sum_{j=1, j \neq i}^{n} A^t(i,j)\sigma_{ij}(t)}, \quad (5)$$

where $\sigma_{ij}(t) = \Sigma(i,j)$ at time t and $A^t(i,i)$ is the (i,i) component of the channel gain matrix at time t. The corresponding channel capacity is then defined as

$$\beta_i(\alpha_i) = \log(1 + \alpha_i). \quad (6)$$

Note that the σ_{ij} term in (5) defines the interference on receiver i from transmitter j in the presence of stochastic noise inputs.

Equation (5) defines the SINR from all other receivers to receiver r_i in the presence of white noise. With a power control input $u(t)$, the allocation policy in (4) can be written as

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + Gdw(t), \quad (7)$$

where $B(t)$ denotes the input matrix and $u(t)$ denotes the power inputs at time t. The evolution of the state covariance matrix under the control input can be written as

$$\dot{\Sigma}(t) = (A(t) - B(t)K(t))\Sigma(t) + \Sigma(t)(A(t) - B(t)K(t))' + GG', \quad (8)$$

where $K(t)$ is the feedback gain. The control problem is to find the optimal control inputs from the set of all admissible control inputs, $\mathcal{U}$ such that $\alpha_i(t)$ is steered to some desired $\alpha_i^d(T)$ at some desired instant T. Thus, the control problem can be formally written as

Problem 1.

$$\begin{aligned} \underset{u \in \mathcal{U}}{\textit{argmin}} \quad & J(u(t)) = \mathbb{E}\left[\int_{t=0}^{T} u(t)'u(t)dt\right] \\ \textit{subject to} \quad & (8) \textit{ and} \\ & \Sigma(0) = \Sigma_0, \; \alpha_i(T) = \alpha_i^d(T). \end{aligned} \quad (9)$$

3. RESULTS

We first establish the relationship between the desired SINR at the desired time T and the state covariance matrix at time T. From (5), it is easy to see that for a desired α_i^d at any instant, there exists a positive definite matrix, Π whose elements satisfy (5). Thus, the constraint $\alpha_i(T) = \alpha_i^d(T)$ in problem 1 can be written as $\Sigma(T) = \Pi$, where $\Sigma(t)$ satisfies the differential equation (8). Problem 1 can thus be rewritten as

Problem 2.

$$\begin{aligned} \underset{u \in \mathcal{U}}{\textit{argmin}} \quad & J(u(t)) = \mathbb{E}\left[\int_{t=0}^{T} u(t)'u(t)dt\right] \\ \textit{subject to} \quad & (8) \textit{ and} \\ & \Sigma(0) = \Sigma_0, \; \Sigma(T) = \Pi, \\ & \alpha_i^d(T) = \frac{A^t(i,i)\Pi(i,i)}{\sum_{j=1, j \neq i}^{n} A^t(i,j)\Pi(i,j)}. \end{aligned} \quad (10)$$

Proposition 1. *The optimal feedback gain $K(t)$ solving Problem 2 is of the form*

$$K(t) = -B(t)'P(t), \quad (11)$$

where $P(t)$ is a differentiable matrix function taking values in the set of $n \times n$ symmetric matrices and satisfies both the following equations:

$$\dot{P}(t) = -A(t)'P(t) - P(t)A(t) + P(t)B(t)B(t)'P(t) \quad (12)$$

$$\dot{\Sigma}(t) = (A(t) - B(t)B(t)'P(t))\Sigma(t) + \Sigma(t)(A(t) - B(t)B(t)'P(t))' + GG' \quad (13)$$

with split boundary conditions

$$\Sigma(0) = \Sigma_0, \; \Sigma(T) = \Pi. \quad (14)$$

Proof. Given $P(t)$ is a symmetric differentiable matrix function, the two end-point marginals Σ_0 and Π are fixed when u varies in $\mathcal{U}$, where $\mathcal{U}$ defines the family of finite energy control inputs. Therefore, the cost function in Problem 1 is equivalent to minimizing over $\mathcal{U}$, the modified index

$$\tilde{J}(u(t)) = \mathbb{E}\left[\int_{t=0}^{T} u(t)'u(t)dt + x(T)'P(T)x(T) - x(0)'P(0)x(0))\right]. \quad (15)$$

We can rewrite equation (15) as

$$\tilde{J}(u(t)) = \mathbb{E}\Big[\int_{t=0}^{T} u(t)'u(t)dt + \int_{t=0}^{T} d(x(t)'P(t)x(t))\Big]. \tag{16}$$

Using Ito's rule, we have

$$d(x(t)'P(t)x(t)) = dx(t)'P(t)x(t) + x(t)'P(t)dx(t) + x(t)'\dot{P(t)}x(t)dt + \int_{t=0}^{T} \text{tr}(P(t)GG')dt. \tag{17}$$

Choose $P(t)$ so that it satisfies the Ricatti equation (12). Using (7) together with (12), we can write

$$\int_{t=0}^{T} u(t)'u(t)dt + \int_{t=0}^{T} d(x(t)'P(t)x(t)) = \int_{t=0}^{T} \|u(t) + B(t)'P(t)x(t)\|^2 dt + \int_{t=0}^{T} \text{tr}(P(t)GG')dt + \int_{t=0}^{T} dw(t)'G'P(t)x(t) + \int_{t=0}^{T} x(t)'P(t)Gdw(t). \tag{18}$$

which in turn implies

$$\mathbb{E}\Big[\int_{t=0}^{T} u(t)'u(t)dt + \int_{t=0}^{T} d(x(t)'P(t)x(t))\Big] = \mathbb{E}\Big\{\int_{t=0}^{T} \|u(t)+B(t)'P(t)x(t)\|^2 dt + \int_{t=0}^{T} \text{tr}(P(t)GG')dt\Big\}. \tag{19}$$

where we have used $\mathbb{E}\{dw(t)\} = 0$, since we have assumed that the increment $dw(t)$ follows a Gaussian distribution with zero mean and covariance $dt\mathbf{I}$. The second term in (18) is invariant over $\mathcal{U}$. Hence the optimal feedback control is of the form $u^*(t) = -B(t)'P(t)x(t)$ with optimal feedback gain $K(t) = -B(t)'P(t)$. Therefore, the state covariance of the system driven by the optimal control input evolves according to the first-order linear differential matrix equation in (13). Also, if (13) satisfies the boundary conditions given in (14), then $u^*(t)$ solves Problem 1. □

The constraints in the evolution of the state covariance are defined by the boundary constraints Σ_0 and $\Sigma(T)$ along with the differential equation constraint in (13). It is important to highlight that (12) shares similarities with a typical Linear Quadratic Regulator (LQR) problem, except for the boundary constraint specified in (14). The boundary value for $P(t)$ in (12) is unspecified, and finding $P(0), P(T)$ that satisfies the boundary constraints in (14) is non-trivial. Also, the indefiniteness of $P(t)$ places the context of our problem outside the standard LQR theory. Furthermore, the Riccati equations (12) and (13) are also coupled, not solely due to the boundary constraints, but also in terms of their dynamic behavior. Finding a closed-form solution to these coupled Riccati equations is challenging Chen et al. (2015b,a); Wakolbinger et al. (1990); Beurling (1960); Jamison (1970); Georgiou and Pavon (2015), and establishing both the existence and uniqueness of solutions for this system of equations is non-trivial. Therefore, for the proof of existence and uniqueness of solutions to the coupled Riccati equations with coupled boundary values, we prove it with the simpler time-invariant system, where we assume A and B are time-invariant and (A, B) is controllable. For the linear time-invariant case, the state covariance evolution in (8) can be written as

$$\dot{\Sigma(t)} = (A - BK(t))\Sigma(t) + \Sigma(t)(A - BK(t))' + GG'. \tag{20}$$

Theorem 1. *The state covariance in (20) is controllable if and only if (A, B) is a controllable pair. Also if (7) is controllable and given any positive definite matrices $\Sigma(0)$, Π satisfying the bounds (14) and an arbitrary $Q \succeq 0$, there is a smooth input, $U(t) = -\Sigma(t)K(t)'$ so that the solution of the forced differential equation*

$$\dot{\Sigma(t)} = A\Sigma(t) + \Sigma(t)A' + BU(t)' + U(t)B' + Q \tag{21}$$

satisfies the conditions $\Sigma(0) = \Sigma_0$ and $\Sigma(T) = \Pi(T)$, where $K(t) = -B'P(t)$.

Proof. Using Proposition 1 and $U(t) = -\Sigma(t)K(t)'$, there is a bijective correspondence between $U(t)$ and $K(t)$. Now, consider the differential equation

$$\dot{\Sigma(t)} = A\Sigma(t) + \Sigma(t)A' + BU(t)' + U(t)B'. \tag{22}$$

Define $R(t) = e^{-At}\Sigma(t)e^{-A't}$ so that (22) can be written as

$$\dot{R(t)} = e^{-At}BU(t)'e^{-A't} + e^{-At}U(t)B'e^{-A't}, \tag{23}$$

which can be written as

$$\dot{R(t)} = e^{-At}BV(t)' + V(t)B'e^{-A't}, \tag{24}$$

where we have used $V(t) = e^{-At}U(t)$. if (A, B) is a controllable pair, then the system

$$\dot{X(t)} = e^{-At}BV(t)', \tag{25}$$

with inputs given by columns of $V(t)$ is controllable since the controllability Grammian

$$C(t) = \int_{0}^{T} e^{-A\tau}BB'e^{-A'\tau}d\tau$$

is invertible. Therefore, we can drive (23) to any final state $R(T) = X(T) + X(T)'$. Conversely, if (A, B) is not a controllable pair, then there is a matrix P such that $Pe^{-At}B = 0$ and $R(t)$ in (24) is not controllable.

We now want to establish the existence of control inputs $u(t)$ so the state distribution defined by the second order moment satisfies the SINR constraint in (9). In other words, we wish to prove the non-emptiness of the set of admissible controls $\mathcal{U}(t)$.

Consider a shift matrix, A of size k.

$$A_k = \begin{bmatrix} 0_{k-1} & I_{k-1} \\ 0 & 0'_{k-1} \end{bmatrix}, B_k = \begin{bmatrix} 0_{k-1} \\ 1 \end{bmatrix} \tag{26}$$

where I_k denotes the identity matrix of size k, and 0_k denotes a column vector of size k with entries zero. For an arbitrary controllable pair, (A, B), there exists a constant K and a vector v such that $(A - BK, Bv)$ is controllable (Heymann's Lemma). Also, K can be chosen such that $A - BK$ has all its eigenvalues at the origin. Hence it is equivalent to a shift matrix. Therefore, we can choose K and v such that, after a similarity transformation $(A - BK, Bv)$ becomes (A_n, B_n). We show by induction that for any $k \times k$ positive semidefinite matrix Q_k, the system

$$\dot{\Sigma(t)} = A_k\Sigma(t) + \Sigma(t)A'_k + B_kU_k(t)' + U_k(t)B'_k + Q_k \tag{27}$$

can be steered between any arbitrary positive definite boundary values. This is true for $k = 1$ since when $k = 1$, all the entries of (27) are scalar and can be written as

$$\dot{\Sigma(t)} = 2U_1(t) + Q_1 \tag{28}$$

and the solution of (27) is given by

$$\Sigma_0 + 2\int_{0}^{T} U_1(\tau)d\tau + Qt > 0 \text{ for all } t. \tag{29}$$

The boundary constraints dictate that

$$\Sigma_0 + 2\int_0^T U_1(\tau)d\tau + QT = \Pi. \tag{30}$$

It is easy to see that (30) can be satisfied with any boundary conditions on $U_1(t)$. As an example, when $\Sigma(t) = e^{h(t)} > 0$ where

$$h(t) = a_0 + b_0 t + \frac{a_T - a_0 - Tb_0}{T^2}t^2 + \frac{Tb_0 + Tb_T - 2a_T + 2a_0}{T^3}t^2(t-T) \tag{31}$$

and

$$\begin{aligned} a_0 &= \log(\Sigma_0),\\ a_T &= \log(\Sigma(T))),\\ b_0 &= (2U_1(0)+Q)/\Sigma_0,\\ b_T &= (2U_1(t_s)+Q)/\Sigma(T)). \end{aligned} \tag{32}$$

The polynomial $h(t)$ satisfies $h_0 = a_0, h(T) = a_T, \dot{h}(0) = b_0, \dot{h}(T) = b_T$. It is easy to verify that $\Sigma(t) = e^{h(t)}$ satisfies $\Sigma(0) = \Sigma_0, \Sigma(T) = \Pi, \dot{\Sigma}(0) = 2U_1(0) + Q, \dot{\Sigma}(T) = 2U_1(T) + Q$. The control input $U_1(t)$ can be computed from (30). We now show that the state covariance can be steered between any two boundary constraints for $k = n-1$ and argue that it is true for $k = n$. Equation (27) for $k = n$ can be written as

$$\Gamma_n\dot{\Sigma(t)}\Gamma_n = \Gamma_n A_n\Sigma(t)\Gamma_n + \Gamma_n\Sigma(t)A_n'\Gamma_n + \Gamma_n Q_n\Gamma_n, \tag{33}$$

where Γ is the projection onto the orthogonal complement of the range of B, i.e., $\Gamma B = 0$ and is given by

$$\Gamma_n = \begin{bmatrix} I_{n-1} & 0_{n-1}\\ 0_{n-1}' & 0 \end{bmatrix} \tag{34}$$

We partition $\Sigma(t)$ as

$$\Sigma(t) = \begin{bmatrix} \Sigma_1(t) & \sigma_2(t)\\ \sigma_2'(t) & \sigma_3(t), \end{bmatrix}$$

where Σ_1 is $(n-1)\times(n-1)$, σ_2 is a column vector, and σ_3 is a scalar. Therefore (33) can be written as

$$\begin{bmatrix} \dot{\Sigma_1}(t) & 0_{n-1}\\ 0_{n-1}' & 0 \end{bmatrix} = M\begin{bmatrix} \Sigma_1(t) & 0_{n-1}\\ \sigma_2'(t) & 0 \end{bmatrix} + \begin{bmatrix} \Sigma_1(t) & \sigma_2(t)\\ 0_{n-1}' & 0 \end{bmatrix}M' + \begin{bmatrix} Q_1 & 0_{n-1}\\ 0_{n-1}' & 0 \end{bmatrix} \tag{35}$$

where Q_1 is the $(n-1)\times(n-1)$ block of Q and

$$M = \Gamma A_n = \begin{bmatrix} 0_{n-1} & I_{n-1}\\ 0 & 0_{n-1}' \end{bmatrix} = \begin{bmatrix} A_{n-1} & B_{n-1}\\ 0_{n-1}' & 0 \end{bmatrix}.$$

Thus, equation (35) is in the form

$$\dot{\Sigma_1}(t) = A_{n-1}\Sigma_1(t) + \Sigma_1(t)A_{n-1}' + G\sigma_2(t)' + \sigma_2(t)G' + Q_1$$

The control input $U(t)$ is given by $\sigma_2(t)$, and the boundary conditions for $U(t)$ are defined by the boundary constraints of $\Sigma(t)$. The entries $\sigma_3(0), \sigma_3(T)$ of $\sigma_3(t)$ are admissible since $\Sigma_0, \Pi > 0$. Therefore, we can always choose a smooth function of $\sigma_3(t)$ for $t \in [0,T]$ such that $\Sigma(t) \succeq 0$. □

In Theorem 1, we show that there exist solutions $\{P(t), \Sigma(t) | 0 \leq t \leq T\}$ that satisfy the coupled Riccati equations in (12) and (13) provided $\Sigma(T) \succ GG'$. Further, we show that the state covariance can be steered between any two boundary constraints given, thus satisfying the coupled boundary constraints in (14). For the time-varying case, we require that the corresponding controllability Grammian is positive definite. Similarly, given that the system is controllable, the set of control inputs $\mathcal{U}$ that steers the SINR to the desired value is non-empty. To compute the optimal control inputs, we formulate the problem as a semidefinite program (SDP), which can be effectively solved using optimization tools like CVX Grant and Boyd (2014) or YALMIP Lofberg (2004) as illustrated below.

3.1 Numerical computation of optimal control:

Define the control input $u(t) = -K(t)x(t)$ so that

$$\begin{aligned} \mathbb{E}\left[\int_{t=0}^T u(t)'u(t)dt\right] &= \int_{t=0}^T \text{trace}(\mathbb{E}[u(t)u(t)'])dt\\ &= \int_{t=0}^T \text{trace}(\mathbb{E}[K(t)x(t)x(t)'K(t)'])dt\\ &= \int_0^T \text{trace}(K(t)\Sigma(t)K(t)')dt. \end{aligned} \tag{36}$$

Define $U(t) = -\Sigma(t)K(t)'$ such that the cost function in (36) can be written as

$$J(u(t)) = \int_0^T \text{trace}(U(t)'\Sigma(t)^{-1}U(t))dt. \tag{37}$$

Thus, equation (37) is jointly convex in $U(t)$ and $\Sigma(t)$. Using $U(t) = -\Sigma(t)K(t)'$, we can write (13) as

$$\begin{aligned} \dot{\Sigma(t)} = A(t)\Sigma(t) + \Sigma(t)A(t)'\\ + B(t)U(t)' + U(t)B(t)' + GG', \end{aligned} \tag{38}$$

which is linear in both $U(t)$ and $\Sigma(t)$. The optimization problem in Problem 2 can thus be written as a semi-definite program:

$$\begin{aligned}
&\text{minimize} && \int_0^T \text{trace}(Y(t))dt && \text{(39a)}\\
&\text{subject to} && \text{equations (38) and} && \text{(39b)}\\
& && \Sigma(0) = \Sigma_0 && \text{(39c)}\\
& && \alpha_i^d(T) = \frac{A^t(i,i)\Sigma^T(i,i)}{\sum_{j=1,j\neq i}^n A^t(i,j)\Sigma^T(i,j)}. && \text{(39d)}\\
& && \begin{bmatrix} Y(t) & U(t)'\\ U(t) & \Sigma(t) \end{bmatrix} \succeq 0, && \text{(39e)}
\end{aligned}$$

where we have used Σ^T to denote $\Sigma(T)$.
The problem can be solved by discretizing 38 and the feedback gain can be recovered as $K(t) = -U(t)'\Sigma(t)^{-1}$.

4. NUMERICAL EXAMPLE

We consider a wireless communication system with n transmitters, $\{t_1, t_2, \cdots t_n\}$ and n receivers, $\{r_1, r_2, \cdots r_n\}$. The stochastic-based approximation for controlling the transmitting powers using the time-varying power dynamics is

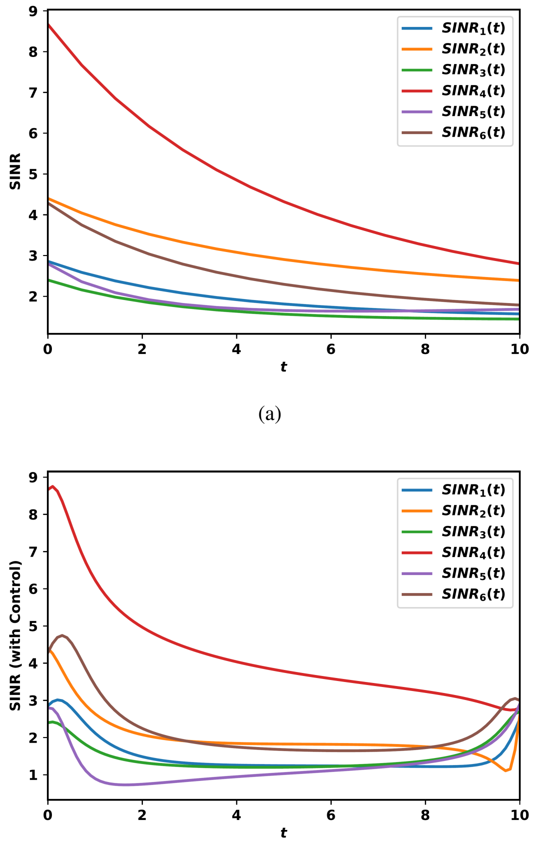
$$\dot{x(t)} = A(t)x(t) + B(t)u(t) + Gdw(t), \tag{40}$$

where $A(t)$ denotes the channel gain matrix and G denotes the noise matrix. We assume that the initial transmitted power fluctuates around its mean with an initial covariance matrix, Σ_0. For illustration, we assume A to be fixed and the values of Σ_0, A, B, and G matrices for simulations are as follows:

$$A = \begin{bmatrix} 0.5 & 0.2 & 0.2 & 0.1 & 0.1 & 0 \\ 0.1 & 0.5 & 0 & 0.4 & 0 & 0 \\ 0.5 & 0.2 & 0.5 & 0.2 & 0 & 0.1 \\ 0.1 & 0 & 0.2 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \end{bmatrix}, \Sigma_0 = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2.2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2.4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2.6 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2.8 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 \end{bmatrix} \quad (41)$$

For all the numerical simulations, we use $dt = 0.01$ for discretizing (13). Figure 2 demonstrates how the stochastic optimal control approach can enhance the efficiency of addressing complex power allocation challenges within communication systems. The desired SINR levels are given as $\alpha_i^d(T = 10) = \{2.5, 2.6, 2.7, 2.8, 2.9, 3\}$ for $i = \{1, 2 \cdots 6\}$ and the optimal required power allocations are computed using (39) with input matrix given in (41).



(a)

(b)

Figure 2. (a) Signal to Interference Plus Noise Ratio curves for a wireless communication system of 6 transmitters and receivers and given channel matrix, A computed using (5). (b) The controlled SINRs achieve the desired values at $t = 10$.

The above application is based on a linear stochastic power allocation model, which is commonly employed in developing power allocation algorithms in Goodman and Mandayam (2000); Foschini and Miljanic (1993, 1995). Another application in the domain of power control is in ensuring efficient and robust power delivery in electrical grids Menara et al. (2022), where the non-linear oscillatory network dynamics model can be reduced in the form of (5) and is widely used in modeling large distribution networks and microgrids Dörfler et al. (2013). The methods described in this study can also be applied to compute the control inputs necessary for achieving both frequency synchronization among electrical grids and a desired power flow pattern.

5. CONCLUSION

In this work, we have proposed a stochastic optimal control approach for controlling power allocation in a plant following Foschini's power allocation technique. The control problem can be translated to the covariance control problem, and the interference among the transmitters can be expressed as a function of the elements in the state covariance matrix. Among the set of all covariance matrices that satisfy the desired SINR, the proposed optimal control approach finds the state covariance that minimizes the cost function defined by the control inputs. Although there exists similarities with the proposed technique and the classical LQG control problem, the solution approach requires solving coupled Riccati equations,presenting a non-trivial computational challenge. We show there exists solutions to the coupled Riccati equations satsfying the terminal bound constraints defined by the desired SINR. Finally, we show the problem can be solved using Semi-definite programming approach.

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