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A Compact Set of Closed-form Equations for the Set of Dubins and Reeds-Shepp Paths

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Abstract: As the demand for robots and vehicles capable of autonomously navigating through obstacle-rich environments increases, so will the need for software capable of achieving such functionality. To find a collision-free trajectory through an obstacle-rich environment, the use of motion planning software may be necessary. When using the kinematic model of a car, the trajectories found by such software would ideally be composed of Dubins or Reeds-Shepp paths, depending on whether reversing is allowed or not. However, current methods for generating these paths tend to be rather verbose. This paper presents a compact set of closed-form equations that describe the set of Dubins and Reeds-Shepp paths. This set is compact in that the equations describe the set of Dubins and Reeds-Shepp path families as opposed to the set of Dubins and Reeds-Shepp path types, as defined in this paper, reducing the complexity of creating motion planning software when using the kinematic model of a car. These equations are derived by modelling the car as an underactuated system on the special Euclidean group in dimension 2 and then solving an associated set of inverse kinematics problems.

Keywords: Robotic systems, Automotive systems.

1. INTRODUCTION

Current methods for generating the set of Dubins paths, presented in Dubins (1957), and Reeds-Shepp paths, first presented in Reeds and Shepp (1990) as a set of 48 paths and later presented in Sussman and Tang (1991) as a set of 46 paths, tend to be rather verbose, such as in Shkel and Lumelsky (2001) in the case of the set of Dubins paths. Limited research has been conducted into deriving a more compact set of closed-form equations that describe these sets of paths.

Following the approach taken in Martínez et al. (2003), a compact set of closed-form equations that describe the set of Dubins and Reeds-Shepp paths can be derived by modelling the car as an underactuated system on the special Euclidean group in dimension 2 and then solving an associated set of inverse kinematics problems. This set is compact in that the equations describe the set of Dubins and Reeds-Shepp path families as opposed to the set of Dubins and Reeds-Shepp path types, as defined in this paper, reducing the complexity of creating motion planning software when using the kinematic model of a car.

Note that, for this paper, all angles are assumed to be in radians.

1.1 Kinematic Model

Let $\{s\}$ denote the spatial frame and $\{b\}$ denote the body frame of a car, where the origin of $\{b\}$ is located at the centre of the rear axle, with its x-axis pointing towards the centre of the front axle and its y-axis pointing towards the centre of the left rear wheel, as shown in Fig. 1.

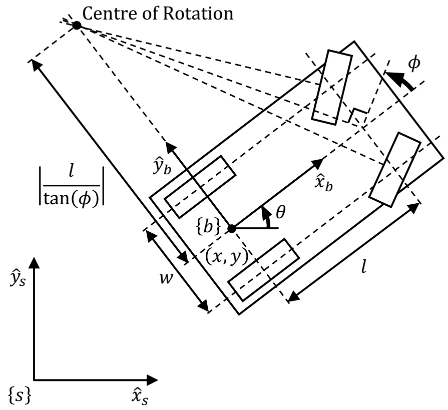

Fig. 1. A car with wheelbase l, track w, steering angle ϕ and body frame $\{b\}$, where (x, y) denotes the position of the origin of $\{b\}$ with respect to the spatial frame $\{s\}$ and θ denotes the orientation of $\{b\}$ with respect to $\{s\}$.

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^2$ denote the set of 2-tuples of real numbers equipped with the dot product, and $\mathbb{S}^1$ denote the 1-dimensional sphere.

Let $\mathbb{R}^2 \times \mathbb{S}^1$ be the configuration manifold of the car and $q \in \mathbb{R}^2 \times \mathbb{S}^1$ denote its configuration, where in coordinates (x, y, θ),

$$q = (x, y, \theta), \tag{1}$$

where $(x, y) \in \mathbb{R}^2$ denotes the position of the origin of $\{b\}$ with respect to the origin of $\{s\}$ and $\theta \in \mathbb{S}^1$ denotes the orientation of $\{b\}$ with respect to $\{s\}$.

Let $l \in \mathbb{R}_{>0}$ denote the wheelbase of the car, $v \in \mathbb{R}$ denote the velocity of the car along the x-axis of $\{b\}$ and $\phi \in [-\phi_{\max}, \phi_{\max}]$ denote the steering angle of the car's front wheels, where $\phi_{\max} \in \left(0, \frac{\pi}{2}\right)$ denotes the maximum steering angle. Therefore,

$$\dot{x} = v\cos(\theta), \tag{2}$$
$$\dot{y} = v\sin(\theta), \tag{3}$$
$$\dot{\theta} = \frac{v\tan(\phi)}{l}. \tag{4}$$

1.2 Motion Primitives

Let $n \in \mathbb{N}$ denote the number of motion primitives along a given path that can be traversed by the car, $SE(2)$ denote the special Euclidean group in dimension 2 and $\mathfrak{se}(2)$ denote the Lie algebra of $SE(2)$.

For $i \in \mathbb{N}_{\leq n+1}$, let

$$q_i = (x_i, y_i, \theta_i) \in \mathbb{R}^2 \times \mathbb{S}^1 \tag{5}$$

denote the configuration of the car such that, for $i \in \mathbb{N}_{\leq n}$,

$$g_{i+1} = g_i \exp(X_i \Delta t_i), \tag{6}$$

where, noting that $\mathbb{R}^2 \times \mathbb{S}^1 \simeq SE(2)$,

$$g_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & x_i \\ \sin(\theta_i) & \cos(\theta_i) & y_i \\ 0 & 0 & 1 \end{bmatrix} \in SE(2) \tag{7}$$

denotes the matrix representation of the configuration of the car,

$$X_i = \begin{bmatrix} 0 & -\omega_i & v_i \\ \omega_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{se}(2) \tag{8}$$

denotes the matrix representation of the planar body twist of the car between g_i and g_{i+1}, and $\Delta t_i \in \mathbb{R}_{\geq 0}$ denotes the duration of the motion from g_i to g_{i+1}, where $v_i \in \mathbb{R}_{\neq 0}$ and

$$\omega_i = \begin{cases} \frac{v_i}{r_i} & \text{if } \phi_i \neq 0, \\ 0 & \text{if } \phi_i = 0, \end{cases} \tag{9}$$

where $\phi_i \in [-\phi_{\max}, \phi_{\max}]$ and

$$r_i = \frac{l}{\tan(\phi_i)}. \tag{10}$$

Therefore,

$$g_{n+1} = g_1 \prod_{i=1}^{n} \exp(X_i \Delta t_i). \tag{11}$$

Note that

$$\exp(X_i \Delta t_i) = \begin{bmatrix} \cos(\Delta\theta_i) & -\sin(\Delta\theta_i) & \Delta x_i \\ \sin(\Delta\theta_i) & \cos(\Delta\theta_i) & \Delta y_i \\ 0 & 0 & 1 \end{bmatrix}, \tag{12}$$

where

$$\Delta x_i = \begin{cases} r_i \sin(\Delta\theta_i) & \text{if } \phi_i \neq 0, \\ v_i \Delta t_i & \text{if } \phi_i = 0, \end{cases} \tag{13}$$

$$\Delta y_i = \begin{cases} r_i\big(1 - \cos(\Delta\theta_i)\big) & \text{if } \phi_i \neq 0, \\ 0 & \text{if } \phi_i = 0, \end{cases} \tag{14}$$

$$\Delta\theta_i = \begin{cases} \frac{v_i \Delta t_i}{r_i} & \text{if } \phi_i \neq 0, \\ 0 & \text{if } \phi_i = 0 \end{cases} \tag{15}$$

denote the changes along the x, y and θ directions of $\{b\}$, respectively. Therefore, for $i \in \mathbb{N}_{\leq n}$,

$$\Delta t_i = \begin{cases} \frac{\Delta\theta_i r_i}{v_i} & \text{if } \phi_i \neq 0, \\ \frac{\Delta x_i}{v_i} & \text{if } \phi_i = 0, \end{cases} \tag{16}$$

where Δx_i and $\Delta\theta_i$ may be determined using (11). Given that g_1 and g_{n+1} are known, it will be beneficial to repose (11) as

$$g_1^{-1} g_{n+1} = \prod_{i=1}^{n} \exp(X_i \Delta t_i), \tag{17}$$

where

$$g_1^{-1} g_{n+1} = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) & \Delta x \\ \sin(\Delta\theta) & \cos(\Delta\theta) & \Delta y \\ 0 & 0 & 1 \end{bmatrix}, \tag{18}$$

where

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{n+1} - x_1 \\ y_{n+1} - y_1 \\ \theta_{n+1} - \theta_1 \end{bmatrix}. \tag{19}$$

For the following sections, let

$$R_1 = \begin{bmatrix} \cos(\Delta\theta_1) & -\sin(\Delta\theta_1) \\ \sin(\Delta\theta_1) & \cos(\Delta\theta_1) \end{bmatrix}. \tag{20}$$

2. DUBINS PATHS

Given a pair of configurations, the shortest path that can be traversed by the car between these configurations without reversing, while ignoring obstacles, belongs to the set of Dubins paths that exist between these configurations. The complete set of Dubins paths is shown in Table 1, where L and R denote motion primitives corresponding to left and right forward motions with a minimum turning radius, respectively, S denotes a motion primitive corresponding to a straight forward motion and C denotes an L or R motion primitive. The necessary conditions for Dubins paths to exist between g_1 and g_4 are shown in Tables 2 and 3. Note that, for this section, for $i \in \{1, 2, 3\}$, $v_i \in \mathbb{R}_{>0}$.

Table 1. Dubins path families and types.

Path family	Path types
CCC	LRL, RLR
CSC	LSL, LSR, RSL, RSR

Table 2. Dubins path turning circle angles.

Path family	$\lvert\Delta\theta_1\rvert$	$\lvert\Delta\theta_2\rvert$	$\lvert\Delta\theta_3\rvert$
CCC	$[0, 2\pi)$	$(\pi, 2\pi)$	$[0, 2\pi)$
CSC	$[0, 2\pi)$	0	$[0, 2\pi)$

Table 3. Dubins path steering angles.

Path family	ϕ_1	ϕ_2	ϕ_3
CCC	$\{-\phi_{\max}, \phi_{\max}\}$	$-\phi_1$	$-\phi_2$
CSC	$\{-\phi_{\max}, \phi_{\max}\}$	0	$\{-\phi_{\max}, \phi_{\max}\}$

2.1 CCC Dubins Paths

For CCC Dubins paths, with reference to (17),

$$\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} = r_1\left(\begin{bmatrix}\sin(\Delta\theta)\\ 1-\cos(\Delta\theta)\end{bmatrix} - 2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right), \tag{21}$$

$$\Delta\theta = \Delta\theta_1 + \Delta\theta_2 + \Delta\theta_3, \tag{22}$$

where

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = R_1\begin{bmatrix}\sin(\Delta\theta_2)\\ 1-\cos(\Delta\theta_2)\end{bmatrix}. \tag{23}$$

Therefore, to generate CCC Dubins paths between g_1 and g_4 using (16), for $i \in \{1,2,3\}$, let

$$\Delta\theta_i = \operatorname{sgn}(\phi_i)\Big(\big(\operatorname{sgn}(\phi_i)\Delta\theta_i'\big) \bmod (2\pi)\Big), \tag{24}$$

where

$$\Delta\theta_1' = 2\arctan\left(\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{\beta}\right) - \frac{\Delta\theta_2'}{2}, \tag{25}$$

$$\Delta\theta_2' = \left(\operatorname{sgn}(\phi_1)\arccos\left(1-\frac{\alpha^2+\beta^2}{2}\right)\right) \bmod (2\pi), \tag{26}$$

$$\Delta\theta_3' = \Delta\theta - \Delta\theta_1' - \Delta\theta_2', \tag{27}$$

where, with reference to (21),

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}\sin(\Delta\theta)\\ 1-\cos(\Delta\theta)\end{bmatrix} - \frac{1}{r_1}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}\right). \tag{28}$$

Note that, for $i \in \{1,2,3\}$,

$$\Delta t_i = \frac{\big(\Delta\theta_i' r_i\big) \bmod \big(2\pi|r_i|\big)}{v_i}. \tag{29}$$

2.2 CSC Dubins Paths

For CSC Dubins paths, with reference to (17),

$$\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} + \begin{bmatrix}r_3\sin(\Delta\theta)\\ r_1 - r_3\cos(\Delta\theta)\end{bmatrix}, \tag{30}$$

$$\Delta\theta = \Delta\theta_1 + \Delta\theta_3, \tag{31}$$

where

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = R_1\begin{bmatrix}\Delta x_2\\ r_3 - r_1\end{bmatrix}. \tag{32}$$

Therefore, to generate CSC Dubins paths between g_1 and g_4 using (16), let

$$\Delta x_2 = \sqrt{\alpha^2+\beta^2-(r_3-r_1)^2} \tag{33}$$

and, for $i \in \{1,3\}$,

$$\Delta\theta_i = \operatorname{sgn}(\phi_i)\Big(\big(\operatorname{sgn}(\phi_i)\Delta\theta_i'\big) \bmod (2\pi)\Big), \tag{34}$$

where, with reference to (30),

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} - \begin{bmatrix}r_3\sin(\Delta\theta)\\ r_1 - r_3\cos(\Delta\theta)\end{bmatrix}, \tag{35}$$

$$\Delta\theta_1' = 2\arctan\left(\frac{\Delta x_2 - \alpha}{\beta + r_3 - r_1}\right), \tag{36}$$

$$\Delta\theta_3' = \Delta\theta - \Delta\theta_1'. \tag{37}$$

Note that, for $i \in \{1,3\}$,

$$\Delta t_i = \frac{\big(\Delta\theta_i' r_i\big) \bmod \big(2\pi|r_i|\big)}{v_i}. \tag{38}$$

2.3 Verification

Fig. 2 shows a set of Dubins paths generated using the equations presented in this section, where $(x_1, y_1), (x_4, y_4) \in \mathbb{R}^2$ are shown in blue and red, respectively.

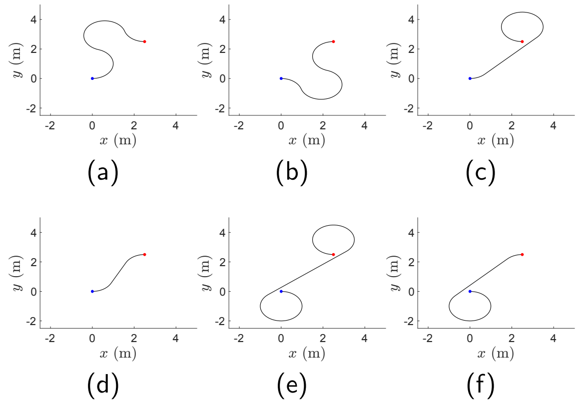

Fig. 2. Dubins paths from $q_1 = (0\text{ m}, 0\text{ m}, 0\text{ rad})$ to $q_4 = (2.5\text{ m}, 2.5\text{ m}, 0\text{ rad})$, where $\phi_{\max} = \frac{\pi}{4}$ and $l = 1$ m. (a) LRL. (b) RLR. (c) LSL. (d) LSR. (e) RSL. (f) RSR.

3. REEDS-SHEPP PATHS

Given a pair of configurations, the shortest path that can be traversed by the car between these configurations, while ignoring obstacles, belongs to the set of Reeds-Shepp paths presented in Sussman and Tang (1991) that exist between these configurations. The complete set of Reeds-Shepp paths presented in Sussman and Tang (1991) is shown in Table 4, where L and R denote motion primitives corresponding to left and right motions with a minimum turning radius, respectively, S denotes a motion primitive corresponding to a straight motion, C denotes an L or R motion primitive, | denotes a change from a forward motion to a reverse motion or vice versa and the + and − superscripts denote the forward and reverse directions of motion, respectively. The necessary conditions for Reeds-Shepp paths to exist between g_1 and g_{n+1} are shown in Tables 5–6.

Table 4. Reeds-Shepp path families and types.

Path family	Path types
$C\|C\|C$	$L^+R^-L^+$, $R^+L^-R^+$
$CC\|C$	$L^+R^+L^-$, $L^-R^-L^+$, $R^+L^+R^-$, $R^-L^-R^+$
$C\|CC$	$L^+R^-L^-$, $L^-R^+L^+$, $R^+L^-R^-$, $R^-L^+R^+$
CSC	$L^+S^+L^+$, $L^+S^+R^+$, $L^-S^-L^-$, $L^-S^-R^-$, $R^+S^+L^+$, $R^+S^+R^+$, $R^-S^-L^-$, $R^-S^-R^-$
$CC\|CC$	$L^+R^+L^-R^-$, $L^-R^-L^+R^+$, $R^+L^+R^-L^-$, $R^-L^-R^+L^+$
$C\|CC\|C$	$L^+R^-L^-R^+$, $L^-R^+L^+R^-$, $R^+L^-R^-L^+$, $R^-L^+R^+L^-$
$C\|CSC$	$L^+R^-S^-L^-$, $L^+R^-S^-R^-$, $L^-R^+S^+L^+$, $L^-R^+S^+R^+$, $R^+L^-S^-L^-$, $R^+L^-S^-R^-$, $R^-L^+S^+L^+$, $R^-L^+S^+R^+$
$CSC\|C$	$L^+S^+L^+R^-$, $L^+S^+R^+L^-$, $L^-S^-L^-R^+$, $L^-S^-R^-L^+$, $R^+S^+L^+R^-$, $R^+S^+R^+L^-$, $R^-S^-L^-R^+$, $R^-S^-R^-L^+$
$C\|CSC\|C$	$L^+R^-S^-L^-R^+$, $L^-R^+S^+L^+R^-$, $R^+L^-S^-R^-L^+$, $R^-L^+S^+R^+L^-$

Table 5. Reeds-Shepp path turning circle angles.

Path family	$\|\Delta\theta_1\|$	$\|\Delta\theta_2\|$	$\|\Delta\theta_3\|$	$\|\Delta\theta_4\|$	$\|\Delta\theta_5\|$
$C\|C\|C$	$[0,\pi]$	$[0,\pi]$	$[0,\pi]$	–	–
$CC\|C$	$[0,\|\Delta\theta_2\|]$	$[0,\frac{\pi}{2}]$	$[0,\|\Delta\theta_2\|]$	–	–
$C\|CC$	$[0,\|\Delta\theta_2\|]$	$[0,\frac{\pi}{2}]$	$[0,\|\Delta\theta_2\|]$	–	–
CSC	$[0,\frac{\pi}{2}]$	0	$[0,\frac{\pi}{2}]$	–	–
$CC\|CC$	$[0,\|\Delta\theta_2\|]$	$[0,\frac{\pi}{2}]$	$\|\Delta\theta_2\|$	$[0,\|\Delta\theta_2\|]$	–
$C\|CC\|C$	$[0,\|\Delta\theta_2\|]$	$[0,\frac{\pi}{2}]$	$\|\Delta\theta_2\|$	$[0,\|\Delta\theta_2\|]$	–
$C\|CSC$	$[0,\frac{\pi}{2}]$	$\frac{\pi}{2}$	0	$[0,\frac{\pi}{2}]$	–
$CSC\|C$	$[0,\frac{\pi}{2}]$	0	$\frac{\pi}{2}$	$[0,\frac{\pi}{2}]$	–
$C\|CSC\|C$	$[0,\frac{\pi}{2}]$	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$[0,\frac{\pi}{2}]$

Table 6. Reeds-Shepp path steering angles.

Path family	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
$C\|C\|C$	$\{-\phi_{\max},\phi_{\max}\}$	$-\phi_1$	$-\phi_2$	–	–
$CC\|C$	$\{-\phi_{\max},\phi_{\max}\}$	$-\phi_1$	$-\phi_2$	–	–
$C\|CC$	$\{-\phi_{\max},\phi_{\max}\}$	$-\phi_1$	$-\phi_2$	–	–
CSC	$\{-\phi_{\max},\phi_{\max}\}$	0	$\{-\phi_{\max},\phi_{\max}\}$	–	–
$CC\|CC$	$\{-\phi_{\max},\phi_{\max}\}$	$-\phi_1$	$-\phi_2$	$-\phi_3$	–
$C\|CC\|C$	$\{-\phi_{\max},\phi_{\max}\}$	$-\phi_1$	$-\phi_2$	$-\phi_3$	–
$C\|CSC$	$\{-\phi_{\max},\phi_{\max}\}$	$-\phi_1$	0	$\{-\phi_{\max},\phi_{\max}\}$	–
$CSC\|C$	$\{-\phi_{\max},\phi_{\max}\}$	0	$\{-\phi_{\max},\phi_{\max}\}$	$-\phi_3$	–
$C\|CSC\|C$	$\{-\phi_{\max},\phi_{\max}\}$	$-\phi_1$	0	$-\phi_2$	$-\phi_4$

Table 7. Reeds-Shepp path velocity directions.

Path family	$\mathrm{sgn}(v_1)$	$\mathrm{sgn}(v_2)$	$\mathrm{sgn}(v_3)$	$\mathrm{sgn}(v_4)$	$\mathrm{sgn}(v_5)$
$C\|C\|C$	1	-1	1	–	–
$CC\|C$	$\{-1,1\}$	$\mathrm{sgn}(v_1)$	$-\mathrm{sgn}(v_2)$	–	–
$C\|CC$	$\{-1,1\}$	$-\mathrm{sgn}(v_1)$	$\mathrm{sgn}(v_2)$	–	–
CSC	$\{-1,1\}$	$\mathrm{sgn}(v_1)$	$\mathrm{sgn}(v_2)$	–	–
$CC\|CC$	$\{-1,1\}$	$\mathrm{sgn}(v_1)$	$-\mathrm{sgn}(v_2)$	$\mathrm{sgn}(v_3)$	–
$C\|CC\|C$	$\{-1,1\}$	$-\mathrm{sgn}(v_1)$	$\mathrm{sgn}(v_2)$	$-\mathrm{sgn}(v_3)$	–
$C\|CSC$	$\{-1,1\}$	$-\mathrm{sgn}(v_1)$	$\mathrm{sgn}(v_2)$	$\mathrm{sgn}(v_3)$	–
$CSC\|C$	$\{-1,1\}$	$\mathrm{sgn}(v_1)$	$\mathrm{sgn}(v_2)$	$-\mathrm{sgn}(v_3)$	–
$C\|CSC\|C$	$\{-1,1\}$	$-\mathrm{sgn}(v_1)$	$\mathrm{sgn}(v_2)$	$\mathrm{sgn}(v_3)$	$-\mathrm{sgn}(v_4)$

3.1 $C|C|C$, $CC|C$ and $C|CC$ Reeds-Shepp Paths

For $C|C|C$, $CC|C$ and $C|CC$ Reeds-Shepp paths, with reference to (17),

$$\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} = r_1\left(\begin{bmatrix}\sin(\Delta\theta)\\ 1-\cos(\Delta\theta)\end{bmatrix} - 2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right), \tag{39}$$

$$\Delta\theta = \Delta\theta_1 + \Delta\theta_2 + \Delta\theta_3, \tag{40}$$

where

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = R_1\begin{bmatrix}\sin(\Delta\theta_2)\\ 1-\cos(\Delta\theta_2)\end{bmatrix}. \tag{41}$$

Therefore, to generate $C|C|C$, $CC|C$ and $C|CC$ Reeds-Shepp paths between g_1 and g_4 using (16), for $i \in \{1,2,3\}$, let

$$\Delta\theta_i = \mathrm{sgn}(v_i)\mathrm{sgn}(\phi_i)\Big(\big(\mathrm{sgn}(v_i)\mathrm{sgn}(\phi_i)\Delta\theta_i'\big) \bmod (2\pi)\Big), \tag{42}$$

where

$$\Delta\theta_1' = 2\arctan\left(\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{\beta}\right) - \frac{\Delta\theta_2'}{2}, \tag{43}$$

$$\Delta\theta_2' = \arccos\left(1 - \frac{\alpha^2+\beta^2}{2}\right), \tag{44}$$

$$\Delta\theta_3' = \Delta\theta - \Delta\theta_1' - \Delta\theta_2', \tag{45}$$

where, with reference to (39),

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}\sin(\Delta\theta)\\ 1-\cos(\Delta\theta)\end{bmatrix} - \frac{1}{r_1}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}\right). \tag{46}$$

Note that, for $i \in \{1,2,3\}$,

$$\Delta t_i = \frac{\Delta\theta_i' r_i}{v_i} \bmod \frac{2\pi|r_i|}{|v_i|}. \tag{47}$$

3.2 CSC Reeds-Shepp Paths

For CSC Reeds-Shepp paths, with reference to (17),

$$\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} + \begin{bmatrix}r_3\sin(\Delta\theta)\\ r_1 - r_3\cos(\Delta\theta)\end{bmatrix}, \tag{48}$$

$$\Delta\theta = \Delta\theta_1 + \Delta\theta_3, \tag{49}$$

where

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = R_1\begin{bmatrix}\Delta x_2\\ r_3 - r_1\end{bmatrix}. \tag{50}$$

Therefore, to generate CSC Reeds-Shepp paths between g_1 and g_4 using (16), let

$$\Delta x_2 = \mathrm{sgn}(v_2)\sqrt{\alpha^2+\beta^2-(r_3-r_1)^2} \tag{51}$$

and, for $i \in \{1,3\}$,

$$\Delta\theta_i = \mathrm{sgn}(v_i)\mathrm{sgn}(\phi_i)\Big(\big(\mathrm{sgn}(v_i)\mathrm{sgn}(\phi_i)\Delta\theta_i'\big) \bmod (2\pi)\Big), \tag{52}$$

where, with reference to (48),

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} - \begin{bmatrix}r_3\sin(\Delta\theta)\\ r_1 - r_3\cos(\Delta\theta)\end{bmatrix}, \tag{53}$$

$$\Delta\theta_1' = 2\arctan\left(\frac{\Delta x_2 - \alpha}{\beta + r_3 - r_1}\right), \tag{54}$$

$$\Delta\theta_3' = \Delta\theta - \Delta\theta_1'. \tag{55}$$

Note that, for $i \in \{1,3\}$,

$$\Delta t_i = \frac{\Delta\theta_i' r_i}{v_i} \bmod \frac{2\pi|r_i|}{|v_i|}. \tag{56}$$

3.3 $CC|CC$ and $C|CC|C$ Reeds-Shepp Paths

For $CC|CC$ and $C|CC|C$ Reeds-Shepp paths, with reference to (17),

$$\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} = r_1\left(\begin{bmatrix}-\sin(\Delta\theta)\\ 1+\cos(\Delta\theta)\end{bmatrix} - 2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right), \tag{57}$$

$$\Delta\theta = \Delta\theta_1 + \Delta\theta_4, \tag{58}$$

where

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = R_1\begin{bmatrix}\sin(\Delta\theta_2)\\ 2-\cos(\Delta\theta_2)\end{bmatrix}. \tag{59}$$

Therefore, to generate $CC|CC$ and $C|CC|C$ Reeds-Shepp paths between g_1 and g_5 using (16), for $i \in \{1,2,4\}$, let

$$\Delta\theta_i = \mathrm{sgn}(v_i)\mathrm{sgn}(\phi_i)\Big(\big(\mathrm{sgn}(v_i)\mathrm{sgn}(\phi_i)\Delta\theta_i'\big) \bmod (2\pi)\Big), \tag{60}$$

where

$$\Delta\theta_1' = \mathrm{sgn}(\phi_1)\arccos\left(\frac{\alpha\sqrt{16-(\alpha^2+\beta^2-5)^2}+\beta(\alpha^2+\beta^2+3)}{4(\alpha^2+\beta^2)}\right), \tag{61}$$

$$\Delta\theta_2' = \arccos\left(\frac{5-\alpha^2-\beta^2}{4}\right), \tag{62}$$

$$\Delta\theta_4' = \Delta\theta - \Delta\theta_1', \tag{63}$$

where, with reference to (57),

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}-\sin(\Delta\theta)\\ 1+\cos(\Delta\theta)\end{bmatrix} - \frac{1}{r_1}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}\right). \quad (64)$$

Note that, for $i \in \{1,2,4\}$,

$$\Delta t_i = \frac{\Delta\theta_i' r_i}{v_i} \bmod \frac{2\pi|r_i|}{|v_i|}. \quad (65)$$

3.4 $C|CSC$ Reeds-Shepp Paths

For $C|CSC$ Reeds-Shepp paths, with reference to (17),

$$\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} = \begin{bmatrix}r_4\sin(\Delta\theta)\\ r_1 - r_4\cos(\Delta\theta)\end{bmatrix} - \begin{bmatrix}\alpha\\ \beta\end{bmatrix}, \quad (66)$$

$$\Delta\theta = \Delta\theta_1 + \Delta\theta_2 + \Delta\theta_4, \quad (67)$$

where

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = R_1\begin{bmatrix}(r_1+r_4)\sin(\Delta\theta_2)\\ 2r_1 - \Delta x_3\sin(\Delta\theta_2)\end{bmatrix}. \quad (68)$$

Therefore, to generate $C|CSC$ Reeds-Shepp paths between g_1 and g_5 using (16), let

$$\Delta x_3 = \operatorname{sgn}(v_3)\sqrt{\alpha^2+\beta^2-(r_1+r_4)^2} + 2r_1\sin\left(\Delta\theta_2'\right) \quad (69)$$

and, for $i \in \{1,2,4\}$,

$$\Delta\theta_i = \operatorname{sgn}(v_i)\operatorname{sgn}(\phi_i)\Big(\big(\operatorname{sgn}(v_i)\operatorname{sgn}(\phi_i)\Delta\theta_i'\big) \bmod (2\pi)\Big), \quad (70)$$

where, with reference to (66),

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}r_4\sin(\Delta\theta)\\ r_1 - r_4\cos(\Delta\theta)\end{bmatrix} - \begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}, \quad (71)$$

$$\Delta\theta_1' = 2\arctan\left(\frac{\Delta x_3 + (\beta - 2r_1)\sin\left(\Delta\theta_2'\right)}{\alpha\sin\left(\Delta\theta_2'\right) + r_1 + r_4}\right), \quad (72)$$

$$\Delta\theta_2' = \operatorname{sgn}(v_2)\operatorname{sgn}(\phi_2)\left(\frac{\pi}{2}\right), \quad (73)$$

$$\Delta\theta_4' = \Delta\theta - \Delta\theta_1' - \Delta\theta_2'. \quad (74)$$

Note that, for $i \in \{1,2,4\}$,

$$\Delta t_i = \frac{\Delta\theta_i' r_i}{v_i} \bmod \frac{2\pi|r_i|}{|v_i|}. \quad (75)$$

3.5 $CSC|C$ Reeds-Shepp Paths

For $CSC|C$ Reeds-Shepp paths, with reference to (17),

$$\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} + \begin{bmatrix}-r_3\sin(\Delta\theta)\\ r_1 + r_3\cos(\Delta\theta)\end{bmatrix}, \quad (76)$$

$$\Delta\theta = \Delta\theta_1 + \Delta\theta_3 + \Delta\theta_4, \quad (77)$$

where

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = R_1\begin{bmatrix}\Delta x_2 + 2r_3\sin(\Delta\theta_3)\\ r_3 - r_1\end{bmatrix}. \quad (78)$$

Therefore, to generate $CSC|C$ Reeds-Shepp paths between g_1 and g_5 using (16), let

$$\Delta x_2 = \operatorname{sgn}(v_2)\sqrt{\alpha^2+\beta^2-(r_3-r_1)^2} - 2r_3\sin\left(\Delta\theta_3'\right) \quad (79)$$

and, for $i \in \{1,3,4\}$,

$$\Delta\theta_i = \operatorname{sgn}(v_i)\operatorname{sgn}(\phi_i)\Big(\big(\operatorname{sgn}(v_i)\operatorname{sgn}(\phi_i)\Delta\theta_i'\big) \bmod (2\pi)\Big), \quad (80)$$

where, with reference to (76),

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} - \begin{bmatrix}-r_3\sin(\Delta\theta)\\ r_1 + r_3\cos(\Delta\theta)\end{bmatrix}, \quad (81)$$

$$\Delta\theta_1' = 2\arctan\left(\frac{\Delta x_2 + 2r_3\sin\left(\Delta\theta_3'\right) - \alpha}{\beta + r_3 - r_1}\right), \quad (82)$$

$$\Delta\theta_3' = \operatorname{sgn}(v_3)\operatorname{sgn}(\phi_3)\left(\frac{\pi}{2}\right), \quad (83)$$

$$\Delta\theta_4' = \Delta\theta - \Delta\theta_1' - \Delta\theta_3'. \quad (84)$$

Note that, for $i \in \{1,3,4\}$,

$$\Delta t_i = \frac{\Delta\theta_i' r_i}{v_i} \bmod \frac{2\pi|r_i|}{|v_i|}. \quad (85)$$

3.6 $C|CSC|C$ Reeds-Shepp Paths

For $C|CSC|C$ Reeds-Shepp paths, with reference to (17),

$$\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} = \begin{bmatrix}-r_4\sin(\Delta\theta)\\ r_1 + r_4\cos(\Delta\theta)\end{bmatrix} - \begin{bmatrix}\alpha\\ \beta\end{bmatrix}, \quad (86)$$

$$\Delta\theta = \Delta\theta_1 + \Delta\theta_2 + \Delta\theta_4 + \Delta\theta_5, \quad (87)$$

where

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = R_1\begin{bmatrix}(r_1+r_4)\sin(\Delta\theta_2)\\ 2r_1 - \big(\Delta x_3 + 2r_4\sin(\Delta\theta_4)\big)\sin(\Delta\theta_2)\end{bmatrix}. \quad (88)$$

Therefore, to generate $C|CSC|C$ Reeds-Shepp paths between g_1 and g_6 using (16), let

$$\Delta x_3 = \operatorname{sgn}(v_3)\sqrt{\alpha^2+\beta^2-(r_1+r_4)^2} + 2\Big(r_1\sin\left(\Delta\theta_2'\right) - r_4\sin\left(\Delta\theta_4'\right)\Big) \quad (89)$$

and, for $i \in \{1,2,4,5\}$,

$$\Delta\theta_i = \operatorname{sgn}(v_i)\operatorname{sgn}(\phi_i)\Big(\big(\operatorname{sgn}(v_i)\operatorname{sgn}(\phi_i)\Delta\theta_i'\big) \bmod (2\pi)\Big), \quad (90)$$

where, with reference to (86),

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}-r_4\sin(\Delta\theta)\\ r_1 + r_4\cos(\Delta\theta)\end{bmatrix} - \begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}, \quad (91)$$

$$\Delta\theta_1' = 2\arctan\left(\frac{\Delta x_3 + (\beta - 2r_1)\sin\left(\Delta\theta_2'\right) + 2r_4\sin\left(\Delta\theta_4'\right)}{\alpha\sin\left(\Delta\theta_2'\right) + r_1 + r_4}\right), \quad (92)$$

$$\Delta\theta_2' = \operatorname{sgn}(v_2)\operatorname{sgn}(\phi_2)\left(\frac{\pi}{2}\right), \quad (93)$$

$$\Delta\theta_4' = \operatorname{sgn}(v_4)\operatorname{sgn}(\phi_4)\left(\frac{\pi}{2}\right), \quad (94)$$

$$\Delta\theta_5' = \Delta\theta - \Delta\theta_1' - \Delta\theta_2' - \Delta\theta_4'. \quad (95)$$

Note that, for $i \in \{1,2,4,5\}$,

$$\Delta t_i = \frac{\Delta\theta_i' r_i}{v_i} \bmod \frac{2\pi|r_i|}{|v_i|}. \quad (96)$$

3.7 Verification

Fig. 3 shows a set of Reeds-Shepp paths generated using the equations presented in this section, where $(x_1, y_1), (x_{n+1}, y_{n+1}) \in \mathbb{R}^2$ are shown in blue and red, respectively.

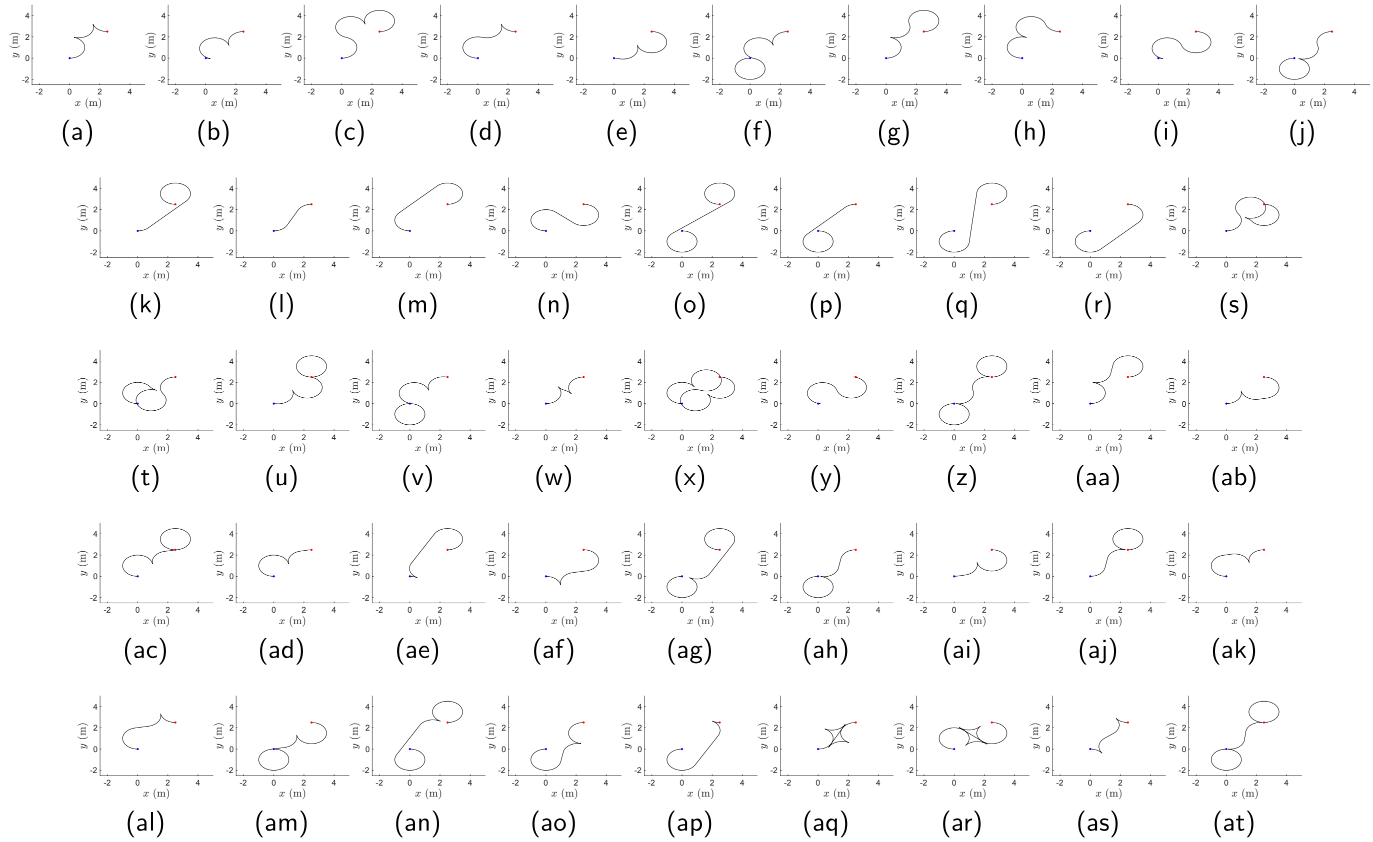

Fig. 3. Reeds-Shepp paths from $q_1 = (0 \text{ m}, 0 \text{ m}, 0 \text{ rad})$ to $q_{n+1} = (2.5 \text{ m}, 2.5 \text{ m}, 0 \text{ rad})$, where $\phi_{\max} = \frac{\pi}{4}$ and $l = 1$ m. (a) $L^+R^-L^+$. (b) $R^+L^-R^+$. (c) $L^+R^+L^-$. (d) $L^-R^-L^+$. (e) $R^+L^+R^-$. (f) $R^-L^-R^+$. (g) $L^+R^-L^-$. (h) $L^-R^+L^+$. (i) $R^+L^-R^-$. (j) $R^-L^+R^+$. (k) $L^+S^+L^+$. (l) $L^+S^+R^+$. (m) $L^-S^-L^-$. (n) $L^-S^-R^-$. (o) $R^+S^+L^+$. (p) $R^+S^+R^+$. (q) $R^-S^-L^-$. (r) $R^-S^-R^-$. (s) $L^+R^+L^-R^-$. (t) $L^-R^-L^+R^+$. (u) $R^+L^+R^-L^-$. (v) $R^-L^-R^+L^+$. (w) $L^+R^-L^-R^+$. (x) $L^-R^+L^+R^-$. (y) $R^+L^-R^-L^+$. (z) $R^-L^+R^+L^-$. (aa) $L^+R^-S^-L^-$. (ab) $L^+R^-S^-R^-$. (ac) $L^-R^+S^+L^+$. (ad) $L^-R^+S^+R^+$. (ae) $R^+L^-S^-L^-$. (af) $R^+L^-S^-L^-$. (ag) $R^-L^+S^+L^+$. (ah) $R^-L^+S^+R^+$. (ai) $L^+S^+L^+R^-$. (aj) $L^+S^+R^+L^-$. (ak) $L^-S^-L^-R^+$. (al) $L^-S^-R^-L^+$. (am) $R^+S^+L^+R^-$. (an) $R^+S^+R^+L^-$. (ao) $R^-S^-L^-R^+$. (ap) $R^-S^-R^-L^+$. (aq) $L^+R^-S^-L^-R^+$. (ar) $L^-R^+S^+L^+R^-$. (as) $R^+L^-S^-R^-L^+$. (at) $R^-L^+S^+R^+L^-$.

4. CONCLUSION

This paper presented a compact set of closed-form equations that describe the set of Dubins and Reeds-Shepp paths. This set is compact in that the equations describe the set of Dubins and Reeds-Shepp path families as opposed to the set of Dubins and Reeds-Shepp path types, as defined in this paper, reducing the complexity of creating motion planning software when using the kinematic model of a car. These equations were derived by modelling the car as an underactuated system on the special Euclidean group in dimension 2 and then solving an associated set of inverse kinematics problems. Together with a path selection and collision detection algorithm, these equations may be used to produce software capable of finding a collision-free trajectory through an obstacle-rich environment when using the kinematic model of a car.

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