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Optimal Decision-Making in a Captive Users Context

A.Y. Kibangou [*,**] **R.C. Kalaoane** [***]

[*] *Univ. Grenoble Alpes, CNRS, Inria, Grenoble INP, GIPSA-Lab, Grenoble, France (e-mail: alain.kibangou@univ-grenoble-alpes.fr)*
[**] *Faculty of Science, University of Johannesburg (Auckland Park Campus), Johannesburg 2006, South Africa*
[***] *Centre for Applied and Research Innovation in the Built Environment (CARINBE), Dept. of Construction Management and Quantity Surveying, University of Johannesburg, Johannesburg 0184, South Africa (e-mail: retsepilek@uj.ac.za)*

Abstract: The scarcity of resources and the structuring of markets for essential resources (water, energy, transportation) lead to a category of users being considered captive. Dissatisfaction among these users can lead to social events with consequences that are difficult to predict. Inspired by compartmental models used in mathematical epidemiology, in this paper, we introduce a new model allowing to analyze dissatisfaction among captive users. The model has a single asymptotically stable equilibrium and its solution is monotone. It is then used for optimal decision-making to avoid reaching a defined critical threshold. In order to take into account the uncertainties of the model parameters, a scenario-based optimization approach is developed.

Keywords: Decision-making, Essential services, Captive users, Dissatisfaction, Scenario optimization.

1. INTRODUCTION

Globally, development agendas and strategies are crafted with the central goal of fostering equitable access to services, which is often regarded as a human right, Koehler (2018). Indeed, equal access to essential services promotes socioeconomic development and ensures social justice, Plata et al. (2019). However, in the realms of modern society, the intricate interplay between infrastructure and consumer dynamics delineates a landscape where certain users find themselves entrapped within essential service markets. This phenomenon, often referred to as captive usage, manifests prominently in transportation, energy, and water sectors, shaping consumption patterns and market dynamics alike, Fang et al. (2021). Understanding the intricacies of captive users within these sectors is paramount, not only for economic and policy formulation but also for ensuring equitable access and sustainable resource management. Transportation, energy, and water infrastructures serve as lifelines, underpinning societal functions and facilitating economic growth. However, the nature of these services often engenders a form of dependence among consumers, compelling them to rely on designated providers within constrained markets. Captive users, thus trapped within these systems, face unique challenges ranging from limited choice and pricing disparities to implications for environmental sustainability and social equity,Rayburn (2015); Rayburn et al. (2020). Additionally, the absence of adequate services have caused dissatisfaction, Furunes and Mkono (2019); Mamokhere (2020), posing a threat to overall well being of captive users. The dissatisfaction of captive users is a pervasive concern. Constrained by limited or nonexistent alternatives, they frequently experience frustration and discontentment with the quality, accessibility, and affordability of the services they depend upon. Whether facing exorbitant pricing, inadequate service provision, or lack of responsiveness to consumer needs, the grievances of captive users underscore systemic flaws and inequities within these sectors. Moreover, disparities in access to essential services exacerbate socioeconomic inequalities, further amplifying the discontentment among marginalized communities. Such a discontentment can turn into a social bomb, the timing and consequences of which are unpredictable. In essence, modeling population dynamics of dissatisfaction enhances comprehension of societal shifts in order to plan interventions, allocate resources effectively, and ultimately work towards creating inclusive and resilient communities. Therefore, the purpose of this paper is to model the population dynamics of dissatisfaction and then, knowing that resources for enhancing quality of service are limited, an optimal control problem is formulated and solved to mitigate such a dissatisfaction. The proposed model is inspired from compartmental models well known in mathematical epidemiology, precisely the SIS (Susceptible Infectious Susceptible). The main idea here is to consider dissatisfaction as an infection that propagates within a constant population.

Historically, compartmental models have been essential to analyze the dynamics of infectious diseases, Kermack and Mc Kendrick (1927); Pimenov et al. (2012); Elhia et al. (2021); Niazi et al. (2021b) and their impact on daily life, Niazi et al. (2021a). Despite having their roots in epidemiology, compartmental models have found use outside of infectious diseases and have been useful tool for understanding complex systems in society such as criminal behaviour, Sooknanan and Seemungal (2023), predicting transport and fate of radionuclides in the marine environment,Maderich et al. (2018), and customer reaction towards digital banking, Méndez-Suárez and Danvila-del Valle (2023), to cite a few. To the best to authors knowledge, this is the first time compartmental models are used to model dissatisfaction in general and in particular within the captive users context. The model is analyzed, the equilibria are explicitly computed and their stability analyzed. In addition, the trajectory of the system can be precisely derived. We use this property to solve an optimal time and magnitude of intervention problem to prevent dissatisfaction going beyond a given threshold. To overcome uncertainties in the model parameters, randomized strategies as in scenario optimization approach, Calafiore and Campi (2006); Campi et al. (2009); Alamo et al. (2009), is adopted.

The paper is organized as follows. The model is described and analyzed in section 2. Various optimization approaches are discussed and evaluated in section 3. Finally concluding remarks and future works are provided in section 4.

2. MODEL DEFINITION AND ANALYSIS

SIS is a compartmental model in which a population can transit between two stages: susceptible and infected. Once infected, they don't get immunity and therefore can go back to the susceptible compartment after they recover from the sickness. The model is constituted with a system of differential equations. Denoting $s(t)$ the fraction of the population that is not yet sick and by $i(t)$ that of infected people. The model is given by

$$\dot{s}(t) = -\beta s(t)i(t) + \gamma i(t),$$
$$\dot{i}(t) = \beta s(t)i(t) - \gamma i(t).$$

Here, $\beta > 0$ represents the infection rate while $\gamma > 0$ stands for the recovery rate. In this model, people can't escape from these two compartments. The model has two equilibrium points: disease free $(s^*, i^*) = (1, 0)$ and $(s^*, i^*) = (\gamma/\beta, 1 - \gamma/\beta)$.

Similar to SIS, we introduce the SDS (Satisfied-Dissatisfied-Satisfied) model as a compartmental model for analysing spreading of dissatisfaction among captive users of a given service. The model is constituted with two compartments: satisfied (S) and dissatisfied (D). Each compartment is characterized by a single state: the proportion $S(t)$ of satisfied users and $D(t)$ that of dissatisfied ones. The whole population being captive, they cannot leave the system. Therefore we adopt a constant population model:

$$S(t) + D(t) = 1 \qquad (1)$$

Transition from one compartment to another results from social connections (word-of-mouth) and positive/negative actions on the provided service. The model is then given by:

$$\dot{S}(t) = \beta S(t)D(t) - f(\mu)S(t) + g(\mu)D(t) \qquad (2a)$$

$$\dot{D}(t) = -\beta S(t)D(t) + f(\mu)S(t) - g(\mu)D(t) \qquad (2b)$$

where:

- β stands for the word-of-mouth (WOM) coefficient. It can be positive or negative. A positive value means that the social interaction leads to increase the number of satisfied users while a negative value stands for spreading of dissatisfaction.
- μ accounts for the quality of service of the considered service while $f(\mu)$ and $g(\mu)$ allow modeling how the quality of service impacts satisfaction and dissatisfaction, respectively. They are assumed to be strictly positive functions.

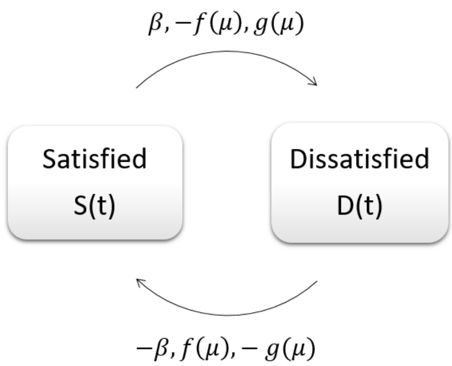

Fig. 1. SDS model's chart.

Given (1), equation (2b) can be rewritten as:

$$\dot{D}(t) = \beta D^2(t) - c(\beta, \mu)D(t) + f(\mu) \qquad (3)$$

with

$$c(\beta, \mu) = \beta + f(\mu) + g(\mu) \qquad (4)$$

or equivalently

$$\dot{D}(t) = \mathcal{P}(D(t)) \qquad (5)$$

with

$$\mathcal{P}(x) = \beta x^2 - c(\beta, \mu)x + f(\mu). \qquad (6)$$

2.1 Analysis of the equilibrium points

In order to carry out stability analysis of the critical points of this autonomous differential equation, we first state some properties of the polynomial $\mathcal{P}(x)$ through few lemmas.

Lemma 1. Since $g(\mu) > 0$, the polynomial $\mathcal{P}(x)$ admits at least one root outside the unit circle.

Proof: For a polynomial $P(z) = a_2z^2 + a_1z + a_0$, the Jury criterion states that the roots are inside the unit circle if and only the following three conditions hold, Jury (1963):

- (i) $a_2 > a_0$
- (ii) $a_2 + a_1 + a_0 > 0$
- (iii) $a_2 - a_1 + a_0 > 0$

For the polynomial $\mathcal{P}(x)$, the second condition corresponds to $\beta - c(\beta, \mu) + f(\mu) > 0$. However, using (4), one gets $\beta - c(\beta, \mu) + f(\mu) = -g(\mu) < 0$. The Jury criterion fails. Therefore, there is at least one root outside the unit circle. □

Lemma 2. Since $f(\mu) > 0$ and $g(\mu) > 0$, the quantity $\Delta = c^2(\beta, \mu) - 4\beta f(\mu)$ is strictly positive.

Proof: Δ can be expanded as
$\Delta = \beta^2 + f^2(\mu) + g^2(\mu) - 2\beta f(\mu) + 2\beta g(\mu) + 2f(\mu)g(\mu)$.
Let us consider first the case $\beta < 0$. We can rewrite Δ as

$$\Delta = (\beta + g(\mu))^2 + f^2(\mu) - 2\beta f(\mu) + 2f(\mu)g(\mu)$$

which is a sum of strictly positive numbers.
Consider now the case $\beta > 0$. We can rewrite Δ as

$$\Delta = (\beta - f(\mu))^2 + g^2(\mu) + 2\beta g(\mu) + 2f(\mu)g(\mu)$$

As for the previous case, Δ is a sum of strictly positive numbers. □

Lemma 3. The polynomial $\mathcal{P}(x)$ admits two roots

$$x_0(\beta,\mu) = \frac{c(\beta,\mu) + \sqrt{\Delta}}{2\beta} \tag{7}$$

$$x_1(\beta,\mu) = \frac{c(\beta,\mu) - \sqrt{\Delta}}{2\beta} \tag{8}$$

which are such that

(1) (i) $0 < x_1(\beta,\mu) < 1 < x_0(\beta,\mu)$ if $\beta > 0$.
(2) (ii) $x_0(\beta,\mu) < 0 < x_1(\beta,\mu) < 1$ if $\beta < 0$

Proof: Since from Lemma 2, $\Delta = c^2(\beta,\mu) - 4\beta f(\mu) > 0$, $\mathcal{P}(x)$ admits the two real valued roots $x_0(\beta,\mu)$ and $x_1(\beta,\mu)$ given by (7) and (8) respectively. We can note that

$$x_0(\beta,\mu) - x_1(\beta,\mu) = \frac{\sqrt{\Delta}}{\beta} \tag{9}$$

and

$$x_0(\beta,\mu) + x_1(\beta,\mu) = \frac{c(\beta,\mu)}{\beta}. \tag{10}$$

Let us consider the case $\beta > 0$. Since $f(\mu) > 0$, the two solutions are strictly positive. From (9), we have $x_1(\beta,\mu) < x_0(\beta,\mu)$. From Lemma 1, we know that at least one root is greater than 1. Then, we can conclude that $x_0(\beta,\mu) > 1$. To prove that $x_1(\beta,\mu)$ is less than one, we can analyze $Q(x_0,x_1) = (x_0(\beta,\mu) - 1)(x_1(\beta,\mu) - 1)$. Expanding it, we get

$$Q(x_0,x_1) = x_0(\beta,\mu)x_1(\beta,\mu) - (x_0(\beta,\mu) + x_1(\beta,\mu)) + 1.$$

Knowing that $x_0(\beta,\mu)x_1(\beta,\mu) = \frac{f(\mu)}{\beta}$ and using (10), we get:

$$Q(x_0,x_1) = \frac{f(\mu) - c(\beta,\mu) + \beta}{\beta} = -\frac{g(\mu)}{\beta} < 0$$

Meaning $Q(x_0,x_1) = (x_0(\beta,\mu)-1)(x_1(\beta,\mu)-1) < 0$. Since $x_0(\beta,\mu) - 1$, we can conclude that $x_1(\beta,\mu) < 1$.

Let us consider the case $\beta < 0$. Since $f(\mu) > 0$, the two solutions are nonzero and have opposite sign. From (9) we can conclude that $x_0(\beta,\mu) < x_1(\beta,\mu)$, meaning that $x_0(\beta,\mu) < 0$ and $0 < x_1(\beta,\mu)$. From Lemma 1, we note that at least one root is outside the unit circle. Assume that $x_1(\beta,\mu)$ is greater than 1. As in the previous case, we can analyze again $Q(x_0,x_1)$. Since $\beta < 0$, we get $Q(x_0,x_1) = -\frac{g(\mu)}{\beta} > 0$. Therefore $x_0(\beta,\mu) - 1 > 0$, which contradicts $x_0(\beta,\mu) < 0$. As a consequence, $x_1(\beta,\mu)$ is necessarily less than 1 and $x_0(\beta,\mu)$ is the root outside the unit circle. □

Based on these properties, we can now analyze the equilibrium points of the SDS model and its trajectory.

Proposition 1. $x_1(\beta,\mu) = \frac{c(\beta,\mu)-\sqrt{\Delta}}{2\beta}$ is the single equilibrium of the SDS model. It is asymptotically stable.

Proof: From (5), we know that the equilibrium points of the system are the roots $x_0(\beta,\mu)$ and $x_1(\beta,\mu)$ of the second order polynomial $\mathcal{P}(.)$. From Lemma 3, we know that $|x_0(\beta,\mu)| > 1$ and $0 < x_1(\beta,\mu) < 1$. The latter is the single admissible equilibrium point since the state is constrained to be positive and less than 1. If $\beta < 0$, $\mathcal{P}(x) > 0$ if $x \in (x_0(\beta,\mu), x_1(\beta,\mu))$ else $\mathcal{P}(x) < 0$. The phase diagram shows that $x_1(\beta,\mu)$ is an attractor while $x_0(\beta,\mu)$ is a repeller. If $\beta > 0$, $\mathcal{P}(x) < 0$ if $x \in (x_1(\beta,\mu), x_0(\beta,\mu))$ else $\mathcal{P}(x) > 0$. The phase diagram shows that $x_0(\beta,\mu)$ is a repeller while $x_1(\beta,\mu)$ is an attractor. We can therefore conclude that whatever the type of WOM ($\beta > 0$ or $\beta < 0$), $x_1(\beta,\mu)$ is the unique admissible equilibrium and it is asymptotically stable. □

2.2 Analysis of the trajectory of the solution

Now, we can analyze the trajectory of the solution of the SDS model and even compute it explicitly.

Proposition 2. The trajectory $D(t)$ of the SDS model is monotone, positive and less than 1 for any positive initial condition $D(0)$ lower than 1.

Proof: In the domain $(0,1)$, the derivative $\dot{D}(t)$ is positive for $D(t) < x_1(\beta,\mu)$ and negative for $D(t) > x_1(\beta,\mu)$. Therefore, if the initial condition is smaller than the equilibrium point, $D(t)$ will be increasing until reaching the equilibrium point. If the initial condition is greater than the equilibrium point, the trajectory will be decreasing towards the equilibrium point. □

Proposition 3. Given a fraction of dissatisfied users $D(\tau)$ at time τ, the fraction $D(t)$, at any time t, of dissatisfied users evolves as

$$D(t) = \frac{\lambda_0(\tau)x_1(\beta,\mu) - \lambda_1(\tau)x_0(\beta,\mu)e^{-\sqrt{\Delta}(t-\tau)}}{\lambda_0(\tau) - \lambda_1(\tau)e^{-\sqrt{\Delta}(t-\tau)}} \tag{11}$$

with $\lambda_i(\tau) = D(\tau) - x_i(\beta,\mu)$, $i = 0,1$, $x_1(\beta,\mu)$ being given by (8) and $x_0(\beta,\mu)$ by (7).

Proof: Since $\Delta = c^2(\beta,\mu) - 4\beta f(\mu) > 0$, we can rewrite $\mathcal{P}(x)$ as $\mathcal{P}(x) = \beta(x - x_0(\beta,\mu))(x - x_1(\beta,\mu))$, $x_0(\beta,\mu)$ and $x_1(\beta,\mu)$. The differential equation (5) can equivalently be written as

$$\frac{\dot{D}(t)}{(D(t) - x_0(\beta,\mu))(D(t) - x_1(\beta,\mu))} = \beta.$$

In the sequel, for the brevity of notation, we skip the dependency of the equilibrium points with respect to β and μ. Resorting to partial fraction decomposition, one can note that

$$\frac{1}{(D(t) - x_0)(D(t) - x_1)} = \frac{\beta}{\sqrt{\Delta}}\left(\frac{1}{D(t) - x_0} - \frac{1}{D(t) - x_1}\right)$$

that yields

$$\frac{\dot{D}(t)}{D(t) - x_0} - \frac{\dot{D}(t)}{D(t) - x_1} = \sqrt{\Delta}$$

Taking the primitive of this equation, one gets

$$\ln\left|\frac{D(t) - x_0}{D(t) - x_1}\right| = \sqrt{\Delta}t + cste$$

cste being a constant. As a consequence,

$$\frac{D(t) - x_0}{D(t) - x_1} = Ke^{\sqrt{\Delta}t}, \quad K \in \Re.$$

In particular

$$\frac{D(\tau)-x_0}{D(\tau)-x_1}=Ke^{\sqrt{\Delta}\tau}$$

which implies

$$K=\frac{D(\tau)-x_0}{D(\tau)-x_1}e^{-\sqrt{\Delta}\tau}.$$

Therefore

$$\frac{D(t)-x_0}{D(t)-x_1}=\frac{D(\tau)-x_0}{D(\tau)-x_1}e^{\sqrt{\Delta}(t-\tau)}$$

Arranging this equation leads to (11). □

Now let us illustrate the properties of this model with an example.

Example 1. We set the parameters $f(\mu)=g(\mu)=0.2$ and illustrate the trajectory of the system for a positive WOM (Fig. 2) and a negative WOM (Fig. 3). In both cases, the combined effects of $f(\mu)$ and $g(\mu)$ can counterbalance the effect of WOM. The equilibrium point is however different. As expected, the trajectory is monotone; decreasing towards the equilibrium point when the initial condition is $D(0)=0.7$ and increasing when $D(0)=0.3$. Here, β only changes the location of the equilibrium point. □

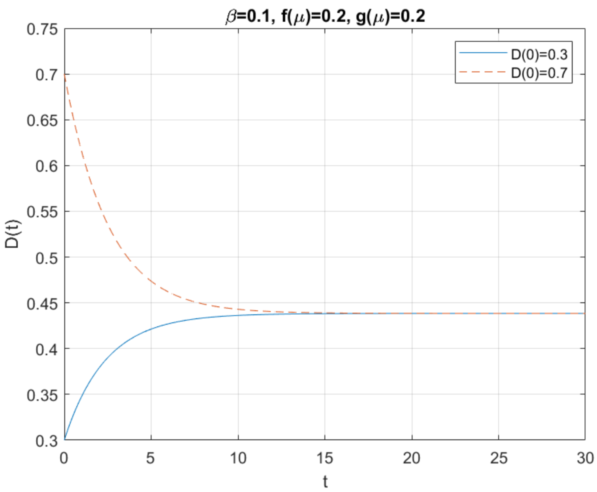

Fig. 2. Dissatisfaction with a positive WOM.

3. OPTIMIZATION PROBLEM

Consider a system with a given dynamics of dissatisfaction. We consider the case were dissatisfaction is growing and We want to avoid reaching a critical threshold of dissatisfaction $\bar{D}$. Let T^* be the instant of intervention of the public decision maker. Therefore, $f(\mu)$ and $g(\mu)$ can be considered as piece-wise functions in order to represent the system before and after the intervention.

$$f(\mu,t)=\gamma H(t)-\gamma(1-e^{-a\mu})H(t-T^*),\quad a>0 \quad (12)$$

and

$$g(\mu,t)=\alpha H(t)+(1-e^{-b\mu})H(t-T^*),\quad b>0 \quad (13)$$

where $H(.)$ stands for the Heaviside step function. With such a choice, γ and α stand for the natural decays of satisfaction and dissatisfaction before intervention. a and b allow taking into account that quality of service can have different impact on satisfaction and dissatisfaction. The

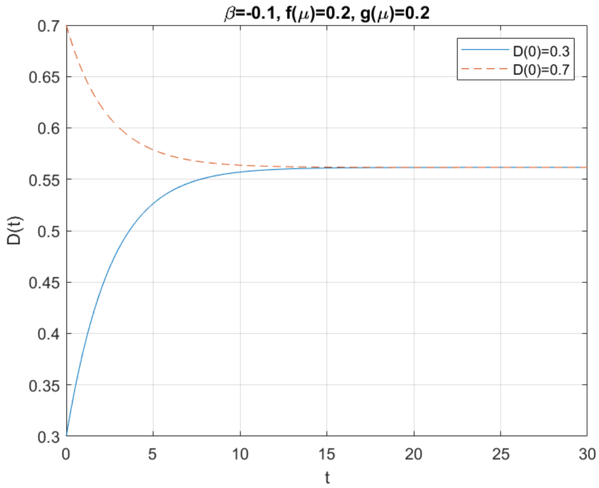

Fig. 3. Dissatisfaction with a negative WOM.

two notions are not symmetric. Indeed, it is well known that a small decrease in quality of service can lead to huge dissatisfaction while a small increase in quality of service will have a small impact on the satisfaction. The trajectory of the dissatisfaction rate is also a continuous piece-wise function defined as follows:

$$D(t)=\underline{D}(t)H(t)+\left(\bar{D}(t)-\underline{D}(t)\right)H(t-T^*) \quad (14)$$

the solutions $\underline{D}(t)$ and $\bar{D}(t)$ being obtained by using the results of the previous section with different initial condition. Continuity of the solution is imposed by constraining the initial condition of the intervention regime to be equal to the last value taken by the dissatisfaction before intervention. Therefore:

$$\underline{D}(t)=\frac{\lambda_0(0)x_1-\lambda_1(0)x_0e^{-\sqrt{\underline{\Delta}}t}}{\lambda_0(0)-\lambda_1(0)e^{-\sqrt{\underline{\Delta}}t}}$$

$$\underline{\lambda}_i(0)=D(0)-\underline{x}_i,\quad i=1,2$$

$$\underline{x}_0=\frac{\beta+\gamma+\alpha+\sqrt{\underline{\Delta}}}{2\beta}$$

$$\underline{x}_1=\frac{\beta+\gamma+\alpha-\sqrt{\underline{\Delta}}}{2\beta}$$

$$\underline{\Delta}=(\beta+\gamma+\alpha)^2-4\beta\gamma$$

$$\bar{D}(t)=\frac{\bar{\lambda}_0(T^*)\bar{x}_1(\mu)-\bar{\lambda}_1(T^*)\bar{x}_0(\mu)e^{-\sqrt{\bar{\Delta}}(t-T^*)}}{\bar{\lambda}_0(T^*)-\bar{\lambda}_1(T^*)e^{-\sqrt{\bar{\Delta}}(t-T^*)}}$$

$$\bar{\lambda}_i(0)=D(T^*)-\bar{x}_i(\mu),\quad i=1,2$$

$$\bar{x}_0(\mu)=\frac{\beta+\gamma e^{-a\mu}+\alpha(1-e^{-b\mu})+\sqrt{\bar{\Delta}}}{2\beta}$$

$$\bar{x}_1(\mu)=\frac{\beta+\gamma e^{-a\mu}+\alpha(1-e^{-b\mu})-\sqrt{\bar{\Delta}}}{2\beta}$$

$$\bar{\Delta}=\left(\beta+\gamma e^{-a\mu}+\alpha(1-e^{-b\mu})\right)^2-4\beta\gamma e^{-a\mu}.$$

Let T be the horizon time, i.e. the time at the end of the intervention. We can assume that the economic normalized cost of the intervention is

$$C_e(\mu,T^*)=\left(1-\frac{T^*}{T}\right)\mu.$$

It can represent subsidies to maintain a service at a given level. A composite cost function as a convex combination of the economic cost and the dissatisfaction rate at the end of the intervention can be adopted:

$$\mathcal{C}_{\nu}(T^*, \mu) = \nu \left(1 - \frac{T^*}{T}\right) \mu + (1 - \nu) D(T), \qquad (15)$$

$D(T)$ being given by (14). The optimization problem to be solved is then formulated as

$$\min_{\mu, T^*} \ \mathcal{C}_{\nu}(\mu, t) \quad s.t. \quad D(t) < D^* \qquad (16)$$

Example 2. In this example we consider a case with a negative WOM characterized by $\beta = -0.5$ while $\gamma = 0.01$ and $\alpha = 0.02$. The social bomb threshold is defined as being equal to $D^* = 0.8$. At time 5.5 the decision maker starts to envision setting up an intervention. Two different cost functions are considered $\mathcal{C}_{3/4}(.)$ and $\mathcal{C}_{1/4}(.)$. The first one gives more importance on the overall cost of the intervention while the second prioritizes the dissatisfaction rate at the end of the intervention. For the functions $f(.)$ and $g(.)$ defined in (12) and (13), we set $a = 2$ and $b = 0.3$. The obtained optimal values of T^* and μ are given in Table 1. The trajectories are depicted in Fig.4.

Table 1. Optimal interventions

Policy	T^*	μ	$\mathcal{C}_e$
Optimal intervention 1: $\mathcal{C}_{3/4}(.)$	7.3	0.29	0.18
Optimal intervention 2: $\mathcal{C}_{1/4}(.)$	7.3	1	0.63

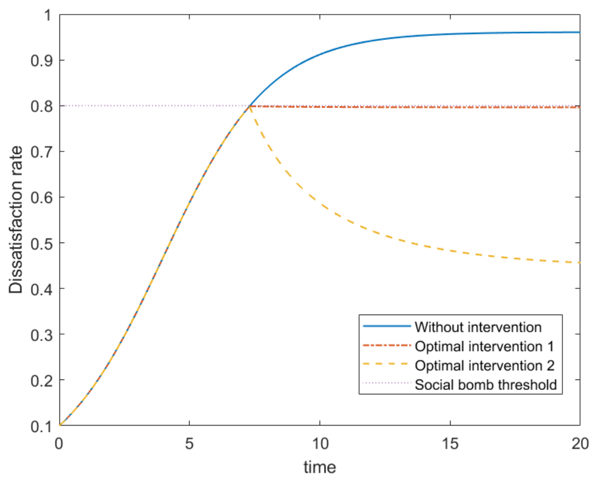

Fig. 4. Dissatisfaction with optimal interventions.

For both cost functions, time for intervention is as late as the dissatisfaction rate is near the social bomb threshold. The first optimal intervention policy minimizes the economic cost so that the dissatisfaction rate remains constantly close to the social bomb threshold. The policy is risky since it relies on a threshold one can't be completely sure of. To overcome this, one can set a lower threshold than the social bomb one. The second policy minimizes the dissatisfaction rate at the end of the intervention period. However, the economic cost is greater than with the first policy. □

The optimization as formulated above requires a perfect knowledge of the parameters β, γ, and α. Estimating these parameters from the observed trajectory is a difficult task. Indeed, such estimation is to be carried out from samples of the transient. Estimated parameters can be far to represent the steady state behavior of the system's dynamics. In what follows, we assume these parameters as being uncertain. Precisely, $\beta \sim \mathcal{U}_{[\underline{\beta}, \bar{\beta}]}$, $\gamma \sim \mathcal{U}_{[\underline{\gamma}, \bar{\gamma}]}$, and $\alpha \sim \mathcal{U}_{[\underline{\alpha}, \bar{\alpha}]}$. Only the bounds are assumed to be known. A robust optimization can be obtained based on the scenario optimization theory, Calafiore and Campi (2006); Campi et al. (2009). The purpose of scenario optimization is to build randomly several, say N, scenarios. Here a scenario is a triplet of parameters (α, β, γ). For each scenario an optimal solution is computed by minimizing a cost function $\mathcal{C}_{\nu}(.)$. Let us denote by (T^*_n, μ_n) the optimal solution obtained from the nth scenario, $n = 1, 2, \cdots, N$, and by $D_{T^*_n, \mu_n}(t)$ the corresponding trajectory. We define by $\mathcal{T}$ a set of test scenarios; each scenario being randomly generated using the defined distributions. Each solution (T^*_n, μ_n) is evaluated in the training set $\mathcal{T}$ and we evaluate the probability to satisfy the social bomb threshold constraint: $\mathbb{P}\left(D_{T^*_n, \mu_n}(t) < D^*\right)$. The optimal solution is then the one maximizing such a probability:

$$n^* = \arg\max_{n} \mathbb{P}\left(D_{T^*_n, \mu_n}(t) < D^*\right). \qquad (17)$$

Among the solutions obtained from each scenario, we select the one such as the social bomb threshold constraint is satisfied with high probability in the test set $\mathcal{T}$.

Example 3. As for the previous example, we consider that the actual values of (α, β, γ) is $(0.02, -0.5, 0.01)$ and $D^* = 0.8$. At time 5.5 the decision maker starts to envision setting up an intervention. However the only knowledge about the parameters (α, β, γ) is the following: $\beta \sim \mathcal{U}_{[-0.7, 0.2}$, $\gamma \sim \mathcal{U}_{[0.01, 0.1]}$, and $\alpha \sim \mathcal{U}_{[0.01, 0.1]}$. $N = 500$ scenarios were generated while 100 scenarios were used for testing. The obtained solutions are given by the table below:

Table 2. Optimal interventions using scenario optimization

Policy	T^*	μ	$\mathcal{C}_e$
Optimal intervention 1: $\mathcal{C}_{3/4}(.)$	6.6	0.42	0.28
Optimal intervention 2: $\mathcal{C}_{1/4}(.)$	6.6	1	0.67

The trajectories of the optimized solutions are depicted in Fig. 5. As in the previous case, the time for intervention is the same for both policies. It is earlier than the one obtained in the previous case. It is sufficiently in advance of the critical threshold. Therefore the policy has a higher economic cost. We can also note that even the first optimal intervention policy (cost function $\mathcal{C}_{3/4}(.)$ allows reducing the dissatisfaction rate.□

4. CONCLUSION

We have introduced SDS (Satisfied-Dissatisfied-Satisfied), a new compartmental model for analyzing spreading of dissatisfaction among captive users of a given essential service. The model has a unique admissible equilibrium point and the trajectory of the solution is monotone. Closed form expression of the equilibrium point has been provided. Therefore by adequately playing with the parameters of the model, one can place the equilibrium at a desired point. The optimal location of the equilibrium results in considering both the economic cost of the intervention and the dissatisfaction rate at the end of the

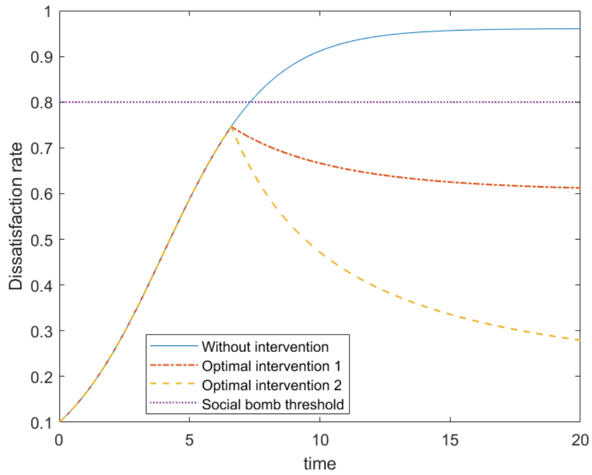

Fig. 5. Dissatisfaction with scenario based optimal interventions.

intervention. It is shown that delaying intervention until the dissatisfaction rate nears the social bomb threshold is optimal. The first policy, aimed at minimizing economic costs while maintaining the dissatisfaction rate close to the social bomb threshold, carries inherent risk due to reliance on a threshold that may not be entirely be certain. On the other hand, the second policy focuses on minimizing the dissatisfaction rate at the end of the intervention period, although at a higher economic cost compared to the first policy. The parameters like WOM being in general unknown, a robust optimization approach is proposed using random scenarios. It allows earlier intervention time. The paper considers a homogeneous population. However, depending on some demographic parameters the reaction to the provided quality of service can differ. In addition, a networked version of this model should be investigated to take into account spatial disparities. This is crucial in particular for territories exhibiting socioeconomic and spatial segregation as it is the case in most developing countries.

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