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Closed-form Approximations for the Minimal Robust Positively Invariant Set using Constrained Convex Generators

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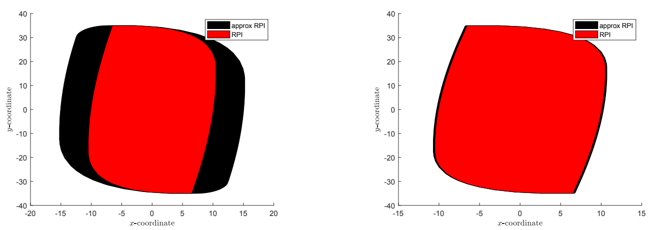
Abstract: Robust Positively Invariant (RPI) sets play a crucial role in constructing terminal constraints for Model Predictive Control (MPC) optimizations and the definition of the invariant sets to be used in Control Barrier Functions (CBFs). However, the solutions in the literature involve either an iterative method that is approaching the true set or an approximation using optimization programs. In this paper, by leveraging the fact that Constrained Convex Generators (CCGs) can represent both polytopes, ellipsoids and other sets, we propose closed-form expressions for outer and inner approximations. Moreover, the tightness can be defined by the system designer based on a straightforward analysis of the norm of the dynamics matrix. We then illustrate how our proposal fairs against the iterative approach highlighting how changing the horizon value influences the added conservatism.

Keywords: Observers for linear systems; Parameter-varying systems; Guidance navigation and control.

1. INTRODUCTION

Reachability tools are a key component of the design of robust Model Predictive Control (MPC) such as in (Bravo et al., 2006), (Langson et al., 2004) and (Mayne et al., 2005). A particular problem of interest is the computation of a Robust Positively Invariant (RPI) set (Rakovic et al., 2005) that can be used to define a terminal constraint such that, when applying a feedback controller, it maintains the system state within the RPI set. However, the computation of the RPI set can also serve to design and check path planning methodologies with obstacle avoidance (Blanchini et al., 2004), check the security of flocking algorithms with collision avoidance techniques (Ribeiro et al., 2020, 2021), and in studying the recursive feasibility of predictive controllers (Mayne et al., 2000). However, in general, these sets must be approximated by iterative algorithms (Rakovic et al., 2005) or by one-step optimizations like (Trodden, 2016) or (Raghuraman and Koeln, 2022).

The computation of the RPI set is linked with that of set-valued state estimation for linear systems. There are various set representations that offer different trade-offs between accuracy and computing time. On the faster spectrum but rather inaccurate we have intervals (Thabet et al., 2014), zonotopes (Combastel, 2003) and ellipsoids (Chernousko, 2005), which add conservatism since intersections do not have a closed-form expression. Other options like polytopes in hyperplane representation (Silvestre et al., 2017b) (Silvestre et al., 2017a), in constrained zonotopes (Scott et al., 2016) format, represented by points (Silvestre, 2022c) or even as A-H polytopes (Sadraddini and Tedrake, 2019) improve on the accuracy with the implicit assumption that disturbances and other unknown signals must belong to a polytope. A generalization named Constrained Convex Generators (CCGs) proposed in (Silvestre, 2022b) has the advantage of being able to represent directly both polytopes, ellipsoids, convex cones, among others and their respective intersections. This set representation will be instrumental for the proposed closed-form expression since it will allow to represent the residual of the approximation by an ℓ_2 ball. The type of sets that CCGs allow to represent are illustrated in Fig. 1, where a toy example for a randomly selected 2×2 dynamics matrix was used and the RPI set was computed with the iterative method with 1000 iterations (a constrained zonotope with 2002 generators) in comparison with the type of set that is being presented (a CCG with 44 generators). The main contributions can be summarized as:



(a) Horizon $H = 20$. (b) Horizon $H = 50$.

Fig. 1. Comparison between the proposed and the solution of the iterative method for computing the RPI set.

- We propose a method that, once a level of conservatism is defined by setting a horizon parameter H,

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has a closed-form expression that can be compiled and used online with a small computing time;
- Leveraging the previous analysis, it is also possible to provide inner and outer approximations through the use of the corresponding approximation for the input set;
- The tightness of the approximations can be computed *a priori* by inspecting the norm of an expression related to the dynamics matrix.

The remainder of the paper is organized as follows. Section 2 formalizes the problem of computing the RPI set, highlighting the main difficulties associated with its infinite description. We review in Section 3 the definition and main set operations for CCGs, while Section 4 describes the proposed approach and implementation details. Simulations are provided in Section 5 with conclusions and directions of future work being given in Section 6.

Notation : We let $\mathsf{0}_n$ denote the n-dimensional vector of zeros and I_n the identity matrix of size n. The operator $\mathrm{diag}(v)$ creates a diagonal matrix with v in the diagonal or extracts the diagonal if the argument is a matrix. The transpose of a vector v is denoted by $v^\intercal$, while the Euclidean norm for vector x is represented as $\|x\|_2 := \sqrt{x^\intercal x}$. On the other hand, $\|x\|_\infty := \max_i |x_i|$. The cartesian product is denoted by $\times$, the Minkowski sum of two sets by $\oplus$ and the intersection after applying a matrix R to the first set by $\cap_R$.

2. ROBUST POSITIVELY INVARIANT SET COMPUTATION

In this paper, we are interested in calculating the RPI set for a discrete-time Linear Time-Invariant (LTI) system that is described by the equation:

$$x(k+1) = Ax(k) + w(k) \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is a strictly stable matrix and $w(k) \in \mathbb{R}^n$. Moreover, in order for the problem to be well-posed, it is necessary that the disturbance signal $w(k)$ be bounded by a compact set containing the origin. The problem is then how to compute a set such that any trajectory will stay inside it regardless of the values for the disturbance signals. More formally, we can retrieve the definition found in (Blanchini, 1999).

Definition 1. (RPI (Blanchini, 1999)). A set $\Omega \subset \mathbb{R}^n$ is a Robust Positively Invariant (RPI) set for the system in (1) if and only if $A\Omega \oplus W \subseteq \Omega$.

In (Rakovic et al., 2005), it is given an iterative solution to find such a set with minimum volume set F_∞ among all possible Ω sets that satisfy Definition 1 by iteratively doing

$$F_\infty = \bigoplus_{i=0}^{\infty} A^i W. \tag{2}$$

Naturally, unless very specific conditions are met regarding the dynamics matrix, one cannot compute exactly the set F_∞. In (Rakovic et al., 2005), the iterative approximation that is proposed is to compute a set denoted by $F(\alpha, s)$ such that $F_\infty \subseteq F(\alpha, s) \subseteq F_\infty \oplus \epsilon B_\infty$ for some user specified error ϵ and B_∞ being the unit ℓ_∞-norm ball. The α parameter should satisfy $A^s W \subseteq \alpha W$. The procedure then iterates over $s = 0, 1, \cdots$ until the desired error ϵ is met, at which point the approximated RPI set is given by:

$$F(\alpha, s) = (1-\alpha)^{-1} \bigoplus_{i=0}^{s} A^i W.$$

As discussed both in (Rakovic et al., 2005) and (Trodden, 2016), the procedure can require thousands of linear programs even for a two state system, depending on how W is represented.

The work in (Trodden, 2016) and (Raghuraman and Koeln, 2022) give a single optimization procedure that is based on the idea of assuming a shape for the RPI set to be calculated and then establishing the containment constraints such that F_∞ is a subset of the solution. The work in (Trodden, 2016) uses a hyperplane format and assumes the normal vectors to the facets of the polytope whereas (Raghuraman and Koeln, 2022) uses the constrained zonotope format and assumes the generator vectors are provided. In this paper, we will go a different route by leveraging the error associated with a norm approximation given that CCGs are capable of representing other sets apart from polytopes.

3. CONSTRAINED CONVEX GENERATORS OVERVIEW

In this section, for the sake of completeness, we recover the CCG definition and main operations that can be found in (Silvestre, 2022b) and (Silvestre, 2022a) in Definition 2 and Definition 3.

Definition 2. (CCGs). A Constrained Convex Generator (CCG) $\mathcal{Z} \subset \mathbb{R}^n$ is defined by the tuple $(G, c, A, b, \mathfrak{C})$ with $G \in \mathbb{R}^{n\times n_g}$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{n_c \times n_g}$, $b \in \mathbb{R}^{n_c}$, and $\mathfrak{C} := \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_{n_p}\}$ such that:

$$\mathcal{Z} = \{G\xi + c : A\xi = b, \xi \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_{n_p}\}.$$

Definition 3. Consider three Constrained Convex Generators (CCGs) as in Definition 2:

- $Z = (G_z, c_z, A_z, b_z, \mathfrak{C}_z) \subset \mathbb{R}^n$;
- $W = (G_w, c_w, A_w, b_w, \mathfrak{C}_w) \subset \mathbb{R}^n$;
- $Y = (G_y, c_y, A_y, b_y, \mathfrak{C}_y) \subset \mathbb{R}^m$;

and a matrix $R \in \mathbb{R}^{m\times n}$ and a vector $t \in \mathbb{R}^m$. The three set operations are defined as:

$$RZ + t = (RG_z, Rc_z + t, A_z, b_z, \mathfrak{C}_z)$$

$$Z \oplus W = \left([G_z \ G_w], c_z + c_w, \begin{bmatrix} A_z & \mathsf{0} \\ \mathsf{0} & A_w \end{bmatrix}, \begin{bmatrix} b_z \\ b_w \end{bmatrix}, \{\mathfrak{C}_z, \mathfrak{C}_w\} \right)$$

$$Z \cap_R Y = \left([G_z \ \mathsf{0}], c_z, \begin{bmatrix} A_z & \mathsf{0} \\ \mathsf{0} & A_y \\ RG_z & -G_y \end{bmatrix}, \begin{bmatrix} b_z \\ b_y \\ c_y - Rc_z \end{bmatrix}, \{\mathfrak{C}_z, \mathfrak{C}_y\} \right).$$

A particular feature that is important for the computations that follow in the next section is the ability of the CCGs to generate multiple other sets like:

- an interval corresponds to $(G, c, [\,], [\,], \|\xi\|_\infty \leq 1)$, for a diagonal matrix G;
- a zonotope is given by $(G, c, [\,], [\,], \|\xi\|_\infty \leq 1)$;
- an ellipsoid is defined by $(G, c, [\,], [\,], \|\xi\|_2 \leq 1)$, for a square matrix G;
- a constrained zonotope or polytope is
$$(G, c, A, b, \|\xi\|_\infty \leq 1);$$
- a convex cone in $\mathbb{R}^n$ is $(G, c, [\,], [\,], \xi \geq 0)$;
- ellipsotopes (Kousik et al., 2022) are given by
$$(G, c, A, b, \|\xi\|_{p_1} \leq 1, \cdots, \|\xi\|_{p_m} \leq 1),$$
for some $p_i > 0$, $1 \leq i \leq m$;
- AH-polytopes (Sadraddini and Tedrake, 2019) are given by $(G, c, [\,], [\,], A\xi \leq b)$.

4. RPI CALCULATION USING CCGS

The RPI computation in this paper draws inspiration from the general expression in (2) and the remark that it could be simplified provided a bound exists for the remainder of the infinite Minkowski sum. We start by rewriting formula (2) in a format amenable to this intuition and using a horizon H as

$$F_\infty = \bigoplus_{i=0}^{H} A^i W \oplus \underbrace{\bigoplus_{i=H+1}^{\infty} A^i W}_{\text{remainder } \Theta}, \tag{3}$$

where the remainder of the horizon Minkowski sums is represented by Θ. As a consequence of (3), if we can provide inner and outer approximations for Θ, given that all operations are done exactly, we will arrive at closed-form expression for sets that serve as an upper and lower bounds.

4.1 Outer Approximation of the RPI set

We will start by looking at the outer approximation as it will build intuition for the inner approximation. In the next theorem, we provide the expression for an outer approximation for a given ℓ_2 ball enclosing set W.

Theorem 1. Let us assume an ℓ_2 ball $\mathcal{B}^u := (\beta I_n, 0_n, [\,], [\,], \|\xi\|_2 \leq 1)$ such that $W \subseteq \mathcal{B}^u$. Then, the CCG given by:

$$\Gamma_H^{\text{outer}} = \bigoplus_{i=0}^{H} A^i W \oplus (\alpha\beta I_n, 0_n, [\,], [\,], \|\xi\|_2 \leq 1)$$

with $\alpha = \sum_{i=1}^{\infty} \|A^{H+i}\|_2$ serves as an outer approximation for F_∞, i.e., $F_\infty \subseteq \Gamma_H^{\text{outer}}$.

Proof. Given the outer approximation $\mathcal{B}^u$ for W, we can compute a point of maximum norm within Θ in (3) as follows:

$$\max_{x\in\Theta} \|x\|_2 = \max_{\substack{x_i \in A^{H+i}W,\\ i\geq H+1}} \left\| \sum_{i=H+1}^{\infty} x_i \right\|_2 .$$

Resorting to the triangular inequality, the maximum norm point satisfies

$$\max_{x\in\Theta} \|x\|_2 \leq \max_{p_1\in W} \|A^{H+1}p_1\|_2 + \max_{p_2\in W} \|A^{H+2}p_2\|_2 + \cdots$$
$$\leq \sum_{i=1}^{\infty} \max_{p_i\in W} \|A^{H+i}p_i\|_2 .$$

Each of the terms $\|A^{H+i}p_i\|_2$ can be further approximated by $\|A^{H+i}\|_2\|p_i\|_2$ leading to

$$\max_{x\in\Theta} \|x\|_2 \leq \sum_{i=1}^{\infty} \max_{p_i\in W} \|A^{H+i}\|_2 \|p_i\|_2$$
$$\leq \sum_{i=1}^{\infty} \|A^{H+i}\|_2 \max_{p\in W} \|p\|_2$$
$$\leq \beta \sum_{i=1}^{\infty} \|A^{H+i}\|_2$$

where we dropped the subscript on p given that all the maximization programs are equivalent. Thus, we can write an approximation Γ_H^{outer} for the RPI set F_∞ based on the definition in (3) as:

$$\Gamma_H^{\text{outer}} = \bigoplus_{i=0}^{H} A^i W \oplus (\alpha\beta I_n, 0_n, [\,], [\,], \|\xi\|_2 \leq 1),$$

with $\alpha = \sum_{i=1}^{\infty} \|A^{H+i}\|_2$. Given that all operations have exact closed-form expressions, the conclusion follows. ■

There are some relevant remarks regarding Theorem 1. First, the ℓ_2 ball that contains W can be formulated as a SemiDefinite Program (SDP) as done in (Hu et al., 2020) for the reachability analysis of closed-loop systems with neural networks as controllers. If W is an interval in n dimensions centered at the origin with maximum side length of L, such a bound is trivial since the point with the largest norm is simply $\sqrt{n}L$, leading to $\mathcal{B}^u$ being

$$(\sqrt{n}LI_n, 0_n, [\,], [\,], \|\xi\|_2 \leq 1).$$

The α value can be computed with the following iterative procedure. Simply run a loop cycle until two consecutive values of the norm have a difference below some numerical threshold and then add the remaining terms until ∞ as the more conservative $\|A\|_2^i$ instead of $\|A^i\|_2$. However, the infinite sum part becomes calculating the sum of a geometric progression with a common ratio given by $\|A\|_2$. Recall that, at some point, these terms are shrinking due to the strictly stable dynamics matrix A.

The expression in Theorem 1 corresponds to H Minkowski sums of polytopes (assuming W is a polytope) with an additional Minkowski sum with an ellipsoid. Moreover, the α computation involves a finite number of matrix products and maintaining a running sum of powers of the norm of A. Therefore, the method has a much better computational complexity in comparison with the option of linear programs used in the literature.

The derivation in Theorem 1 and the assumption of a ℓ_2 ball of radius β for the set W can lead to a different set approximating the true RPI. Consider a point p in the outer bound. Then, we would have

$$\Theta \approx \left\{ \sum_{i=1}^{\infty} A^{H+i}p, p \in \mathcal{B}^\ell \right\} \subseteq \left\{ \beta \sum_{i=H+1}^{\infty} A^i p, \|p\|_2 \leq 1 \right\}.$$

However, the sum is much easier to compute given that

$$\sum_{i=H+1}^{\infty} A^i = (I_n - A)^{-1} - \sum_{i=0}^{H} A^i,$$

resulting in having to compute H matrix multiplications, H matrix additions and 1 matrix inverse. Using this approach, it results in an approximation CCG $\tilde{F}_\infty$ written as:

$$\bigoplus_{i=0}^{H} A^i W \oplus \left(\beta \left[(I_n - A)^{-1} - \sum_{i=0}^{H} A^i \right], 0_n, [\,], [\,], \|\xi\|_2 \leq 1 \right).$$

In the simulations sections, we will see that such an approach results in a set that more closely resembles the true F_∞.

4.2 Inner Approximation of the RPI set

Having presented the outer approximation for the RPI set, we can recover that the ℓ_2 norm of a matrix matches the largest singular value. So in essence, Theorem 1 is translating that an outer approximation for the remainder set corresponds to *enlarging* the ball set $\mathcal{B}^u$ with the maximum expansion factor that results from the linear map with A. Therefore, we can leverage that interpretation to provide an inner approximation by considering an inner set $\mathcal{B}^\ell$ for W and computing the set resulting from applying the smallest singular value.

Lemma 1. Let us assume an ℓ_2 ball $\mathcal{B}^\ell := (\beta I_n, 0_n, [], [], \|\xi\|_2 \le 1)$ such that $\mathcal{B}^\ell \subseteq W$. Then, the CCG given by:

$$\Gamma_H^\ell = \bigoplus_{i=0}^{H} A^i W \oplus \Big(\sum_{i=H+1}^{\infty} \sigma_{\min}(A^i)\beta I_n, 0_n, [], [], \|\xi\|_2 \le 1 \Big)$$

with $\sigma_{\min}(A)$ being the minimum singular value of A. The set Γ_H^ℓ satisfies $\Gamma_H^\ell \subseteq F_\infty$.

Proof. Given that $\mathcal{B}^\ell \subseteq W$, we also have that :

$$\bigoplus_{i=H+1}^{\infty} A^i \mathcal{B}^\ell \subseteq \Theta.$$

Moreover, for any point p we also have that $\|\sigma_{\min}(A^i) I_n p\|_2 \le \|A^i p\|_2$ and, thus, the conclusion follows. ■

An important remark regarding this lemma is that computing the infinite summation for the approximation set is not problematic as with the outer approximation. In this case, if we stop the summation at any given index, the resulting Minkowski sum will be a subset of the set that would be obtained if the summation was carried out until infinity. As done with the previous subsection, we can also investigate what happens if we fix the same point in each of the terms of the Minkowski sum corresponding to Θ. However, given that we are aiming at an inner approximation, we are able to provide the following result.

Theorem 2. For some CCG set W, the CCG given by:

$$\Gamma_H^{\text{inner}} = \bigoplus_{i=0}^{H} A^i W \oplus \left[(I_n - A)^{-1} - \sum_{i=0}^{H} A^i \right] W$$

satisfies $\Gamma_H^{\text{inner}} \subseteq F_\infty$.

Proof. The set Θ can be written explicitly as

$$\left\{ \sum_{i=1}^{\infty} x_i : x_i = A^{H+i} p_i, p_i \in W, \forall i = 1, 2, \cdots \right\}.$$

Fixing the same vector p for all terms of the Minkowski sum results in a subset such that:

$$\left\{ \sum_{i=1}^{\infty} x_i : x_i = A^{H+i} p, p \in W \right\} \subseteq \left\{ \sum_{i=1}^{\infty} x_i : x_i = A^{H+i} p_i, p_i \in W, \forall i = 1, 2, \cdots \right\}.$$

Therefore, we also have that:

$$\left\{ \sum_{i=1}^{\infty} x_i : x_i = A^{H+i} p, p \in W \right\} = \left\{ \sum_{i=1}^{\infty} A^{H+i} p, p \in W \right\}.$$

However, the set on the right-hand side corresponds to the definition of a linear map applied to the original set W, which after the same reformulation of the infinite sum results in

$$\left[(I_n - A)^{-1} - \sum_{i=0}^{H} A^i \right] W,$$

and the conclusion follows. ■

In the next section, we highlight how the proposed closed-form approximations relate to the RPI set obtained through the iterative method and point out the data structure sizes required to store the sets.

5. SIMULATIONS

In this section, the proposed methods for finding inner and outer approximations of the Minimal Robust Positively Invariant (RPI) set for Linear Time Invariant (LTI) systems are illustrated. Simulations were run in Matlab R2021b running on a Fujitsu machine with a Intel Core i7-10510U CPU @ 1.80GHz and 16 GB of memory resorting to the implementation in `https://github.com/danielmsilvestre/ReachTool` where plots are drawn using the overloaded method plot of Yalmip with Mosek as the underlying solver. Figures and code can be found in `https://github.com/danielmsilvestre/RobustPositivelyInvariantPaper`.

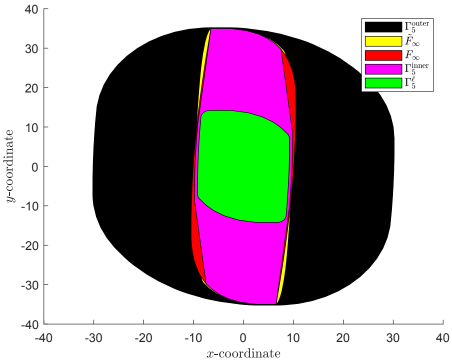

Fig. 2. Comparison between the approximations presented in this paper for a horizon $H = 5$ and a randomly selected A matrix with eigenvalues given by 0.8 and 0.9.

For the first simulation, the disturbance set used is an interval centered at zero corresponding to $[-2, 2] \times [-2, 2]$ but written in CCG format. Since matrix A must be strictly stable and fast eigenvalues are going to result in better approximations, we have fixed the eigenvalues to be 0.8 and 0.9 and select the eigenvectors by sampling each entry uniformly on the the interval $(0, 1)$. The results for a horizon $H = 5$ are illustrated in Figure 2. Interestingly, the outer RPI Γ_5^{outer} is rather conservative along the direction of the smaller singular value. We recover that the theoretical analysis exposed the bound being constructed by applying the largest singular value to the entire set and disregarding the inherent structure in the SVD decomposition. Nonetheless, it is noticeable that the outer approximation is tight along the singular vector associated with the maximum singular value. The set depicted in yellow represents an approximation that may be desirable provided there is no requirement in enforcing it to be an outer or inner bound since the yellow set has points that are outside of the RPI but also does not include the entire RPI set. Lastly, using the minimum singular value is very conservative and can be used whenever there is the need to use an approximation that is strictly inside the RPI.

Another important aspect is the size of the set representation required for each bound. Assuming that we run the iterative algorithm until the maximum norm of a point in $A^i p, p \in W$ is below the threshold of 10^{-6}, we would get a CCG with 294 generators (there are no constraints since the set was an interval). On the other hand, the approximation sets reproduced in Figure 2 require 14 generators.

From Figure 2, it is of relevance to understand how both Γ_H^{inner} and Γ_H^{outer} evolve with H. Figure 3 shows the evolution of the Γ_H^{outer} when we vary $H \in \{5, 10, 12, 40\}$. As expected, the required horizon H is rather small since the remainder set Θ is reducing in size exponentially fast as we increase H.

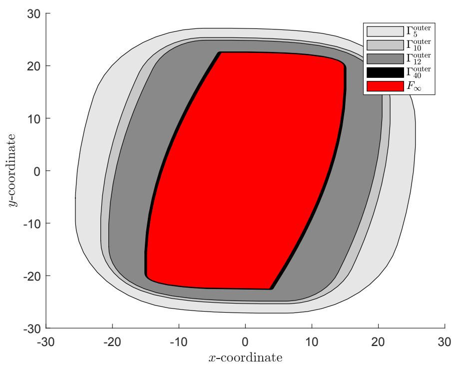

Fig. 3. Variation of the outer approximation Γ_H^{outer} when $H \in \{5, 10, 12, 40\}$.

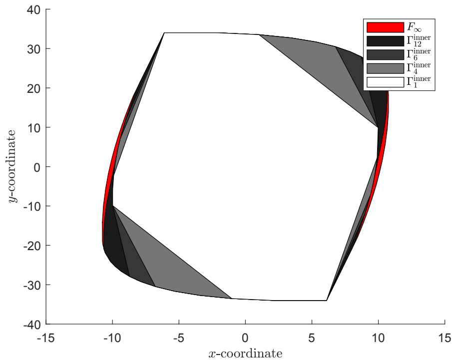

Fig. 4. Variation of the inner approximation Γ_H^{inner} when $H \in \{1, 4, 6, 12\}$.

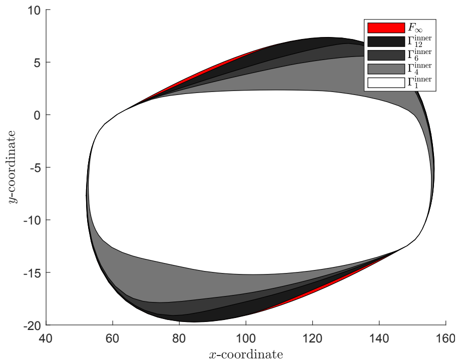

Fig. 5. Variation of the inner approximation Γ_H^{inner} when $H \in \{1, 4, 6, 12\}$ for a generic disturbance set W.

Contrarily to the outer approximation in Figure 3, the set Γ_H^{inner} approaches the true RPI much faster for this example. At $H = 12$, the set shown in black covers almost the entire RPI set. However, since we are still using a rather simple disturbance set, it is of importance to test the approximation in a more complicated structure.

A second simulation was conducted with W having matrix $G \in \mathbb{R}^{2\times 20}$, $c \in \mathbb{R}^{2\times 1}$, $A \in \mathbb{R}^{10\times 20}$, $b \in \mathbb{R}^{10\times 1}$ and $\mathcal{C}$ being a ℓ_∞ unit ball for the first 10 ξ variables and a ℓ_2 unit ball for the remainder. The entries of all matrices and vectors in W were selected from a standard Gaussian distribution. The dynamics matrix was still generated in a similar fashion as the previous simulation.

In Figure 5, it is shown the same evolution for the set Γ_H^{inner} with very similar results as the case of a simple interval. Set

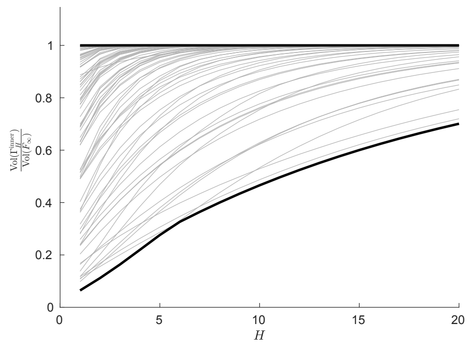

Fig. 6. Ratio of the volumes of the RPI set computed with the iterative procedure versus the inner approximation Γ_H^{inner} as $H \in [1, 20]$ for 100 pairs of random generic disturbance set W and random LTI systems generated using drss.

W has round facets caused by the addition of the ℓ_2 bounds on the generator variables. For this more complicated set W, the advantage of the proposed methodology becomes clearer in comparison with the iterative method. Still maintaining the threshold that the norm bound of the term $A^i W$ be smaller than 10^{-6}, the RPI computed by the iterative method had 3120 generators and 1560 constraints whereas $\Gamma_{12}^{\text{inner}}$ had 280 and 140, respectively. Calculating the RPI using the iterative method took 0.66 s whereas the inner approximation for $H = 12$ took 0.0011 s.

The above results point towards the conclusion that inner approximations following the proposed technique are very efficient since we can use a minimal number of actual Minkowski sums in the horizon (similar to the iterative method) but then compensate the remainder to improve the accuracy that would otherwise require a large number of extra steps of the procedure.

A last point of relevance is to understand the relationship between the hyper-volume of the true RPI in comparison with the inner approximation. In Figure 6, it is depicted the evolution of the ratio between the volumes of the RPI and its inner approximation when considering 100 pairs of random generic sets generated as in the previous simulation and a random system created with drss function from Matlab. As expected, increasing the horizon reduces the difference between the two sets. Moreover, the ratio increases exponentially with systems having larger eigenvalues being the most problematic.

The eigenvalues of the dynamics matrix also play a critical role in the amount of iterations and horizon required for low-conservatism solution. Figure 7 depicts the volume of the sets computed using convhull function of Matlab that uses vertex tessellation on the output of the overloaded plot method from Yalmip. This serves as a metric of how efficient it is to solve optimization problems with the RPI as constraint set as Yalmip computes hundreds of those problems to find vertices. There is a clear advantage as for a horizon $H = 20$ it is taking 5.87 s in comparison with 9.34 s when using the RPI from the iterative method.

6. CONCLUSIONS AND FUTURE WORK

In this paper, we have addressed the problem of computing inner and outer approximations for the minimal Robust Positively Invariant (RPI) set. This problem has a notable application in the design of Control Barrier Functions, terminal constraints for Model Predictive Control and assessing security of a state feedback controller. The current

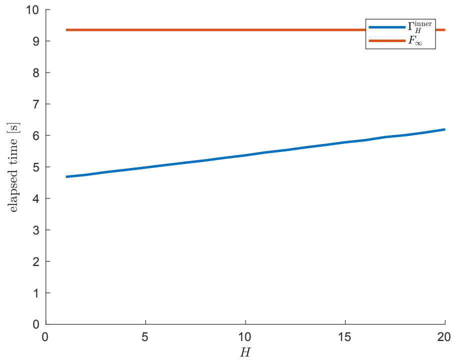

Fig. 7. Average elapsed time over the 100 pairs of random disturbance set and dynamical system for the volume computation for the RPI obtained using the iterative method against the inner approximation Γ_H^{inner} as $H \in [1, 20]$.

proposal is founded on the idea of splitting the traditional computation in two components: a sequence of Minkowski sums to get the general shape of the set and an extra sum with an ellipsoid or with the affine transformation of the disturbance set with the result of the power series of the dynamics matrix for the outer and inner sets, respectively.

The accuracy and performance is illustrated in simulation with the proposed method being able to capture the shape of the RPI even for small horizon values. These results enable future research into the enforcement of feasibility of Model Predictive Controllers, Collision Avoidance for dynamical systems and others where the RPI set must be used within the context of an optimization problem.

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