Preprints of Control Conference Africa 2024
Balaclava, Mauritius, 16-17 September 2024

First outlook on a finite gain from $\mathcal{L}_1$ to $\mathcal{L}_\infty$ space ⋆

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Abstract: In this paper, we present a first outlook on a finite gain from the $\mathcal{L}_1$ to the $\mathcal{L}_\infty$-space. We consider this choice as an interesting gain to research due to its intuitive interpretation in compressor systems used in the manufacturing industry, where it is of utmost importance to maintain a minimum pressure in the system. We start by giving a definition of such a gain and initial results when such a gain exists. We compare this gain to the more commonly used $\mathcal{H}_2$-gain and include simplified examples where the $\mathcal{H}_2$-gain would lead to a more conservative bound. We end the paper with an outlook on the system and the problem description of interest for our future research.

Keywords: Systems theory, Manufacturing plant control, Input-to-State Stability

1. INTRODUCTION

Compressed air in industry

Compressed air systems are used in a wide range of industries and for varying purposes. In the food industry it may be used for dehydration cleaning and packaging, where as in the lumber industry it can be used for sawing and clamping. Additionally, compressed air is used as process gas in petroleum and chemical industry. Beyond these examples, compressed air is used for powering tools, conveyors and actuators in the above mentioned industries and several others, (Saidur et al., 2010). Supplying tools and actuators with compressed air often necessitates a minimal operational pressure to be maintained, as a pressure below this threshold can lead to actuator faults, (Nehler, 2018). Due to the additional safety, reduced tool weight and significantly lower initial and maintenance cost, compared to electric devices, compressed air is projected to be further used as utility throughout industries in the future. The provision of compressed air totals 10 % of industrial electricity use in the EU, resulting in 65,000 GWh/a (Unger and Radgen, 2018). Hence, reducing power consumption by small margins yields a substantial leverage. Lowering the overall system pressure lowers leakage throughout the network and leads to favourable compressor operation, as indicated by Boyle's law ($p_1V_1 = p_2V_2$). For instance, a reduction of 0.7 bar has been identified to save 2.6 - 4 % of power consumption in compressed air generation (Saidur et al., 2010).

Qualitatively, this observation holds true for structurally similar use cases, such as gas or liquid transportation networks. The authors in Nazif et al. (2010) find that lowering the network pressure in urban water distribution while guaranteeing the minimum level, reduces leakage by 30 % for the considered metropolis.

Control of compressor systems

The control of large-scale compressor stations is commonly hierarchically separated into sub-tasks that operate on different time-scales, see Paparella et al. (2013); Xenos et al. (2015). Compressor activation determines the basis that proceeding control agents can act upon. Due to high inertia of movable parts, e.g., bull gear, as well as relevant wear-and-tear and start-up energy consumption, this sub-task is conducted on a low frequency. Energy-optimal scheduling approaches have been proposed in recent literature for chemical plants (Xenos et al., 2015; Kopanos et al., 2015), natural gas pipelines (Zhao et al., 2021; Sun et al., 2020; Paparella et al., 2013; Mahlke et al., 2010) and heating ventilation and air conditioning systems (Uddin et al., 2021; Wong et al., 2022; J. Luo, 2022). In highly volatile demand domains such as manufacturing, compressors are commonly switched reactively according to current demand. At a higher frequency, active compressors with variable outputs, e.g., (multistage) centrifugal compressors, are controlled to stabilize the network's pressure level. This can be broken down further into several sub-tasks. Load sharing, i.e., allocating fractions of total demand to individual compressors, is subject of present-day research throughout industries. Explicitly addressing the split between active machines allows to solve this problem energy-optimally. Recent literature investigates the online correction of model-plant mismatch, (Al Zawaideh et al., 2022; Zagorowska et al., 2023). Natural gas pipelines are represented disproportionately (Zhang et al., 2022; Zagorowska et al., 2020; Peyrl and Cortinovis, 2015; Li et al., 2021; Kumar and Cortinovis, 2017; Cortinovis et al., 2016; Al Zawaideh et al., 2021; Ahmed et al., 2022), although, the authors stress the applicability to other domains that generate compressed gas. The generic compressor station case has been studied in Al Zawaideh et al. (2022); Milosavljevic et al. (2020); Rodrigues (2022). To mitigate model and measurement noise, network stabilization control, e.g., PI(D), assures the pressure level

⋆ This research has been partially funded by the German Federal Ministry of Education and Research (BMBF) through grant 01IS17052 Software Campus 2.0 (Technische Universität Berlin) as part of the Software Campus project 'OCIS'. Corresponding author: M. Axelson-Fisk. Contact: {magnus.axelson-fisk, g.lassahn, knorn}@tu-berlin.de. M. Axelson-Fisk and G. Lassahn have contributed equally to this research.

is maintained by proposing corrections to the compressor actuators. In this paper we focus on this control sub-task and assume the activation schedule to be available. Maintaining centrifugal compressors at a (hardware-wise) safe operation point w.r.t. choke and surge line can either be enforced in the respective Programmable Logic Controller (PLC) by clipping critical control actions or via active surge control (Torrisi et al., 2017). This approach aims at increasing surge margins by applying offset-free nonlinear MPC to mitigate disturbances and model-plant mismatch. An extensive discourse on low-level compressor control is given in Bloch (2010).

Guaranteeing operation safety, i.e., maintaining a critical pressure level at all times is non-negotiable in most industrial applications. Dropping below such a level can severely impede the productivity of individual consumers, potentially disturbing entire production flows. In some cases, even irreversible hardware damage to sensors or membranes is caused. Comprehensively, safety margins are often set well above the critical pressure to ensure the operational constraint. Apart from pressure level, the ratio of energy consumption to provided normalized volume flow (per compressor or compressor station) can be leveraged to judge compressor station performance. However, this measure does not account for additional compressed air demand, caused by increased leakage. For this reason, the pressure safety margin above the critical system pressure should be considered as well when assessing the compressor station performance, as an increased pressure leads to higher leakage.

Mixed-norm gains

The performance of control systems is usually measured by the norm of control costs or the norm of deviations of the output or states from their desired values. Ideally, this performance measure should make use of physically meaningful norms, chosen based on the task. A task-specific selection of norms would be desirable and allow for more accurate and tighter bounds on the effects of disturbances as they reflect better the underlying purpose of the performance measure. The use of the $\mathcal{L}_2$-norm to bound and describe inputs and outputs of a system is common, as it may represent the energy of the signals. Additionally, there are mathematically convenient transformations, such as Parseval's theorem, between time-domain and frequency domain for signals in the $\mathcal{L}_2$-space, leading to the $\mathcal{L}_2$-norm and $\mathcal{L}_2$-gain having been thoroughly explored for linear, see for instance Skogestad and Postlethwaite (2001), as well as non-linear systems. In contrast, for systems with accumulating dynamics, e.g., fill level in fluid containers, charge in buffers, this representation does not translate (intuitively) to a meaningful measure. Here, resource consumption is more directly accounted for by $\mathcal{L}_1$, as briefly touched in Kousoulidis and Forni (2022). This norm emphasises accumulation of input or disturbance, rather than instantaneous variations. Similar arguments can also be found in Briat (2013), in which the author provides characterisations for the $\mathcal{L}_1$- and $\mathcal{L}_\infty$-gains. Interestingly, the results of (Briat, 2013) indicate that the lower bound for the $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain coincides with bound on the $\mathcal{L}_\infty \to \mathcal{L}_\infty$-gain.

Instantaneous or short-lived deviations between states, or outputs, to their desired value is immediately captured by the $\mathcal{L}_\infty$. This property makes the norm especially suitable for the considered pressure network use case, where maximum deviations are of importance.

As we consider systems in which the external inputs are withdrawing compressed air, used in tools which may not run if the pressure drops too low, it is suitable to consider gains and mappings between inputs in the $\mathcal{L}_1$-space and outputs in the $\mathcal{L}_\infty$-space, what we will denote as a $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain. The similar problem of the generalized $\mathcal{H}_2$ problem presented in Rotea (1993) results in a finite gain in the closed-loop from $\mathcal{L}_2 \to \mathcal{L}_\infty$. Though this problem may result in a stabilizing controller in the $\mathcal{H}_2$ space it is not certain that the performance in the loop from $\mathcal{L}_1 \to \mathcal{L}_\infty$ is the same. During our research we have not found any result in control system literature regarding such gains and mappings from $\mathcal{L}_1 \to \mathcal{L}_\infty$. This may be explained by a focus on gains and mappings $\mathcal{L}_2 \to \mathcal{L}_2$, $\mathcal{L}_2 \to \mathcal{L}_\infty$ or $\mathcal{L}_\infty \to \mathcal{L}_\infty$, due to mathematical convenience of moving to and from Hardy-spaces or by utilizing dissipativity or Lyapunov theory. However, due to its intuitive and suitable interpretation the gain and mappings from $\mathcal{L}_1 \to \mathcal{L}_\infty$ are appropriate for both the application at hand, and similar problems.

Contributions

In this paper we present and advocate for a novel gain for mappings $\mathcal{L}_1 \to \mathcal{L}_\infty$. The main contribution can therefore be considered the early result and accompanying examples, which we believe thoroughly motivates for further research of this gain. The results presented in this paper exploit classical properties and proofs of Bounded-Input Bounded-Output, BIBO, stability to give a sufficient condition for linear systems to exhibit such a gain from $\mathcal{L}_1 \to \mathcal{L}_\infty$.

Notation

We denote vector and matrix norms with $|\cdot|_p$ and use $\|\cdot\|_p$ to denote the $\mathcal{L}_p$-norms, which will be defined below.

2. PREREQUISITES

For completion, we briefly define the $\mathcal{L}_p^n$-spaces and their respective norms below, if obvious from context we will omit n.

Definition 1. *The $\mathcal{L}_p^n$-space consists of all measurable functions $f : [0, \infty) \to \mathbb{R}^n$ which satisfy*

$$\int_0^\infty |f(t)|^p dt < \infty. \tag{1}$$

Further, we equip the $\mathcal{L}_p$ spaces with their respective norm

$$\|f\|_p = \left(\int_0^\infty |f(t)|^p dt \right)^{1/p}. \tag{2}$$

We follow this with a definition of the extended $\mathcal{L}_{pe}$-spaces and truncated signals as below.

Definition 2. *The extended $\mathcal{L}_{pe}$-space consists of all measurable truncated functions $u_T : [0, T] \to \mathbb{R}^n$ with $\|u_T\|_p < \infty$, which take the values*

$$u_T(t) = \begin{cases} u(t), & \text{if } 0 \le t \le T \\ 0, & \text{elsewhere.} \end{cases} \tag{3}$$

As we will use Hölder's inequality in this section, we briefly restate the inequality and refer the reader to (Kôsaku, 1995, Ch. 3) for a proof.

Lemma 1 (Hölder's inequality). *Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in \mathcal{L}_p^n$ and $g \in \mathcal{L}_q^n$. Further, define $fg := [f_1 g_1, \ldots, f_n g_n]^{\mathrm{T}}$. Then, Hölder's inequality states*

$$\int_0^\infty |fg| dt \le \left(\int_0^\infty |f|^p dt \right)^{1/p} \left(\int_0^\infty |g|^q dt \right)^{1/q}. \quad (4)$$

3. MAIN RESULTS

3.1 Definition $\mathcal{L}_1 \to \mathcal{L}_\infty$ gain

In general, we consider systems with some input-output mapping

$$y = Hu, \quad (5)$$

where $u : [0, \infty) \to \mathbb{R}^m$ with a bounded $\mathcal{L}_1$ norm, i.e.,

$$\|u\|_1 = \int_0^\infty |u(t)| dt < \infty.$$

As we consider non-linear systems and the relationship between inputs and outputs, we will give a general definition of a $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain. However, for most of the remainder of this paper, we will limit ourselves to LTI-systems.

Definition 3. *A mapping $H : \mathcal{L}_1^m \to \mathcal{L}_\infty^n$ is said to have a finite $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain if there exists finite $\gamma > 0$ and $\beta > 0$ such that for any input in the extended $\mathcal{L}_1^m$-space $u \in \mathcal{L}_{1e}^m$*

$$\|Hu\|_\infty \le \gamma \|u\|_1 + \beta. \quad (6)$$

With Definition 3 in place we start with an example of a mapping with finite $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain. Consider the causal linear convolution operator $y = Hu$ given as

$$y(t) = \int_0^t h(t - \tau) u(t) d\tau, \quad (7)$$

with $u(t) \in \mathcal{L}_{1e}^1$. By altering a proof for the classical BIBO property we can now present the following Lemma.

Lemma 2. *Given a system defined as in (7) there exists a $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain if the impulse response $h(t)$ itself has a bounded $\mathcal{L}_\infty$-norm, $\|h\|_\infty < \infty$. Further, this gain is bounded from above by $\|h\|_\infty$.*

Proof. Inspired by (Khalil, 2002, Ch. 5) we start with

$$\begin{aligned}
|y(t)| &\le \int_0^t |h(t-\tau) u(\tau)| d\tau = \int_0^t |h(\tau) u(t-\tau)| d\tau \\
&\le \int_0^t |h(\tau)||u(t-\tau)| d\tau \\
&= \int_0^t |u(t-\tau)|^{1/q} |u(t-\tau)|^{1/p} |h(\tau)| d\tau \\
&\le \left(\int_0^t |u(t-\tau)| d\tau \right)^{1/q} \left(\int_0^t |u(t-\tau)||h(\tau)|^p d\tau \right)^{1/p}
\end{aligned}$$

Where the last inequality has been obtained by using Hölder's inequality in Lemma 1 with $f = |u(t-\tau)|^{1/q}$ and $g = |u(t-\tau)|^{1/p} |h(\tau)|$. Since $u(t) \in \mathcal{L}_{1e}$ we have $u(t) = 0$ for all $t > T$, this in turn gives us

$$\begin{aligned}
(\|y_T\|_p)^p &\le \int_0^T \|u_T\|_1^{p/q} \int_0^t |u(t-\tau)||h(\tau)|^p d\tau \, dt \\
&= \|u_T\|_1^{p/q} \int_0^T \int_0^t |u(t-\tau)||h(\tau)|^p d\tau dt.
\end{aligned}$$

By reversing the order of the integral we have

$$\begin{aligned}
(\|y_T\|_p)^p &\le \|u_T\|_1^{p/q} \int_0^T |h(\tau)|^p \int_\tau^T |u(t-\tau)| dt d\tau \\
&\le \|u_T\|_1^{p/q} \|u_T\|_1 \|h_T\|_p^p.
\end{aligned}$$

Which, given Lemma 1 and $p/q + 1 = p$, gives us the relationship

$$\|y\|_\infty \le \|h\|_\infty \|u\|_1.$$

This can also be extended for vector-valued convolutions by using the induced matrix ∞-norm. □

Above we utilized that the convolution is commutative to arrive at the result. On the other hand, the classical BIBO-stability condition, $\|h\|_1 < \infty$, leads to $\mathcal{L}_p$-stability, i.e.,

$$\|y\|_p \le \|h\|_1 \|u\|_p.$$

Though a system fulfilling the BIBO stability property would guarantee that for a signal $u(t) \in \mathcal{L}_\infty$ the output will also have a limited $\mathcal{L}_\infty$-norm, this is, for the application of interest, not a suitable performance measure. This is motivated by the difficulty of bounding the amplitude of the disturbances in a meaningful manner. For instance, as tools are swapped an instantaneous high volume flow will occur, which would translate to a large $\mathcal{L}_\infty$-norm, but if the duration is instead kept short it would translate into a lower $\mathcal{L}_1$-norm and the effects are limited.

3.2 Relation to $\mathcal{H}_2$-gain

Another approach to ensure finite gains from $\mathcal{L}_1$ to $\mathcal{L}_\infty$ could be to bound the $\mathcal{L}_1$-norm of a signal by the $\mathcal{L}_2$-norm. This would allow to use the $\mathcal{H}_2$-norm to ensure a finite gain from $\mathcal{L}_1 \to \mathcal{L}_\infty$. However, in this section we will show that this may not be possible for all signals and may result in more conservative results, when possible. For signals satisfying the following relationship

$$\|u\|_2 \le \|u\|_1 \quad (8)$$

a bounded $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain can be ensured by

$$\|y\|_\infty \le \gamma \|u\|_2 \le \gamma \|u\|_1,$$

in which γ could be found as the $\mathcal{H}_2$-gain. However, (8) does not necessarily hold for all signals in $\mathcal{L}_1$ and $\mathcal{L}_2$. As an example when (8) does not hold, we define a truncated signal in the extended $\mathcal{L}_p$-space as

$$u_T(t) = \begin{cases} 2t, & 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

which gives

$$\begin{aligned}
\|u_T\|_1 &= \int_0^T |u(t)| dt = \int_0^T 2t dt = T^2, \text{ and} \\
\|u_T\|_2 &= \left(\int_0^T 4t^2 dt \right)^{1/2} = \frac{2}{\sqrt{3}} T^{3/2}
\end{aligned}$$

Such that for $T = 1$, $\|u_T\|_1 = 1 < \|u_T\|_2 = \frac{2}{\sqrt{3}}$.

For signals which satisfy (8) it would be possible to use the $\mathcal{H}_2$-gain. However, this gain may be significantly larger than the gain found in Lemma 2, as we will see below. A system has a finite $\mathcal{H}_2$-gain from $\mathcal{L}_2 \to \mathcal{L}_\infty$ if

$$\|H\|_{\mathcal{H}_2}^2 = \frac{1}{2\pi} \int_{-\infty}^\infty \operatorname{tr}\left(H(j\omega)^* H(j\omega) \right) d\omega < \infty, \quad (9)$$

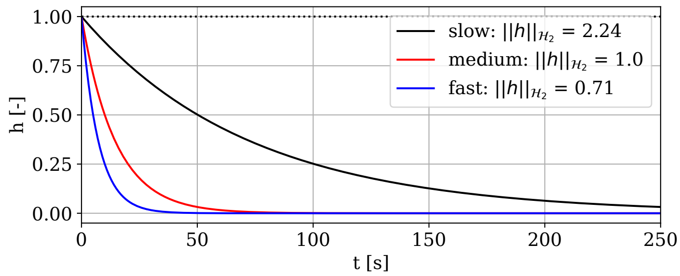

Fig. 1. Impulse responses for systems (13) with the respective $||h||_{\mathcal{H}_2}$. $||h||_\infty = 1$ is illustrated as dashed line.

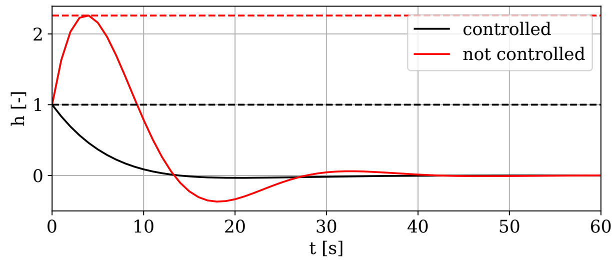

Fig. 2. Impulse responses for system in (16) with and without control. Respective $||h||_\infty$ displayed as broken line. $||h_{\text{controlled}}||_\infty = 1$, $||h_{\text{not controlled}}||_\infty = 2.26$.

where $H(j\omega)$ denotes the Laplace transform of the impulse response of the closed-loop system, (Skogestad and Postlethwaite, 2001, Ch. 9.3). By using Parseval's theorem, this may be translated into

$$\begin{aligned}\|H\|^2_{\mathcal{H}_2} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_i\sum_j h^2_{ij}(t)dt \\ &= \frac{1}{2\pi}\sum_i\sum_j\int_{-\infty}^{\infty} h^2_{ij}(t)dt < \infty.\end{aligned}$$

From this we can deduce that the $\mathcal{H}_2$-gain may be conservative in comparison to the gain defined in Lemma 2. For instance, consider an impulse response defined as

$$h(t) = \begin{cases}1, & 0 \le t \le 4\pi \\ 0, & \text{elsewhere.}\end{cases} \tag{10}$$

The $\mathcal{H}_2$-gain of this impulse response is $\|h\|_{\mathcal{H}_2} = \sqrt{2}$, whereas $\|h\|_\infty = 1$. Thus, for such a system there exists signals $u(t)$, with $\|u\|_2 \le \|u\|_1$, where using the $\mathcal{H}_2$-gain to estimate the $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain would give

$$\|y\|_\infty \le \sqrt{2}\|u\|_2 \le \sqrt{2}\|u\|_1, \tag{11}$$

but Lemma 2 would be less conservative, as

$$\|y\|_\infty \le \|u\|_1 \le \sqrt{2}\|u\|_1. \tag{12}$$

To further illustrate how the $\mathcal{H}_2$-gain may give a conservative bound on the gain from $\mathcal{L}_1$ to $\mathcal{L}_\infty$ for some applications, we consider structurally identical systems with slow, medium and fast dynamics

$$\Sigma : \begin{cases}\dot{x} &= \begin{pmatrix}-1 & 0 \\ 1 & a\end{pmatrix} x + \begin{pmatrix}0 \\ 1\end{pmatrix} u \\ y &= \begin{pmatrix}0 & 1\end{pmatrix} x,\end{cases} \tag{13}$$

with $a \in \{-0.1, -0.5, -1\}$. Fig 1 displays the impulse responses h for each system defined in (13), as well as the respective $\mathcal{H}_2$-gains. While $||h||_\infty$ is 1 for all systems, $||h||_{\mathcal{H}_2}$ increases for slower system dynamics.

3.3 Limitations

We can also note some unfortunate but interesting limitations with Lemma 2. For a general linear system with feedback control and state-space representation as

$$\begin{cases}\dot{x} = (A + BKC)x + B_d d, \\ y = Cx,\end{cases} \tag{14}$$

we can see that Lemma 2 has some disadvantages, which have to be addressed in future research.

Now, $y(t)$ can for this linear system be described by

$$y(t) = Ce^{(A+BKC)t}x_0 + C\int_0^t e^{(A+BKC)(t-\tau)}B_d d(\tau)d\tau.$$

If we assume $x(0) = x_0 = 0$, Lemma 2 gives that it is sufficient to design K such that $\|Ce^{(A+BKC)t}B_d\|_\infty \le \gamma$ for all $0 \le t \le T$, where γ is the prescribed performance. Since the system in (14) is linear and asymptotically stable, we have exponential stability and the following holds

$$\|Ce^{(A+BKC)t}B_d\|_\infty \ge |CB_d|_\infty, \tag{15}$$

where $|CB_d|_\infty$ is the induced matrix-norm.

Thus, we see that the gain described in Lemma 2 is bounded from below by the properties of the system, regardless of the controller. Future research will therefore focus on finding other methods to bound the $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain, such that it is less dependent on the system structure.

As an example of a feedback controller that achieves a $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain less than or equal to one, we examine the system

$$\Sigma : \begin{cases}\dot{x} &= \underbrace{\begin{pmatrix}-1 & -1 \\ 1 & 0\end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix}1 \\ 0\end{pmatrix}}_{B} u + \underbrace{\begin{pmatrix}1 \\ 0\end{pmatrix}}_{B_d} d \\ y &= \underbrace{\begin{pmatrix}1 & 4\end{pmatrix}}_{C} x.\end{cases} \tag{16}$$

By designing the control law $u = -10Cx$ we can reduce the $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain of the system. The impulse responses for the uncontrolled system and the controlled case are given in Fig 2. For this system, we also see that

$$|CB_d| = 1,$$

which gives that this is the lowest bound on the $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain achievable according to Lemma 2.

4. COMPRESSOR NETWORK AND DISTURBANCES

As our future research will focus on the control of an air compression system used in the manufacturing industry this section is devoted to a first definition of the system design and structure. The system at hand is comprised of a bank of compressor configured in a parallel interconnection. To give some insights, we will hence give a system definition and further explain the benefits of seeking controllers that achieve a prescribed gain from disturbances in $\mathcal{L}_1$ to a predefined limit on the pressure in the $\mathcal{L}_\infty$ space.

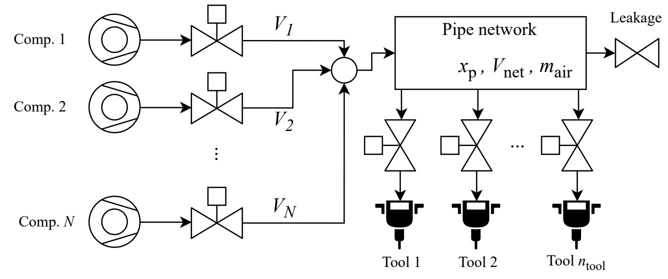

Fig. 3. Schematic of pressurized air network in manufacturing.

4.1 Piping system and disturbances

Let x_p denote a pressure level of a pipe network with volume $\mathcal{V}_\mathrm{net}$, as illustrated in Fig. 3. Through unavoidable leakages, substance continuously streams from within the network to the environment. The amount of leakage is determined by the physical piping and increases with x_p. Additionally, an unknown number of consumers (tools) draws a varying amount of substance from the network, which is used for productive work.

Assuming ideal gas and uniform ambient conditions throughout the plant, a mass balance around the network gives

$$\dot{x}_\mathrm{p} = f_\mathrm{p}\left(x_\mathrm{p}, \sum_{i=1}^{N} y_i, V_\mathrm{out}, d\right) \quad (17)$$

$$= \frac{x_\mathrm{p}}{\mathcal{V}_\mathrm{net}}\left[\sum_{i=1}^{N} y_i - \left(\tilde{V}_\mathrm{out} + d\right)\right], \quad (18)$$

Here, we denote individual volume flow contributions as compressor outputs y_i. We define the disturbance d as the deviation from the expected outflow $\tilde{V}_\mathrm{out}$:

$$V_\mathrm{out} = \tilde{V}_\mathrm{out} + d. \quad (19)$$

Though forecasting future $\tilde{V}_\mathrm{out}$ is of importance for the performance of the compressor system, namely activation scheduling, the prediction is out of the scope of this paper, but is assumed to be available. See Xenos et al. (2015); Kopanos et al. (2015) for an elaborate discussion on this.

We bound d in the extended $\mathcal{L}_1$-space, that is

$$\|d_T\|_1 \le C < \infty. \quad (20)$$

T denotes the safety-critical compressor ramp-up time. As the compressors include a saturation it is necessary to define the finite upper bound C further. This describes the maximum permitted prediction error with respect to the $\mathcal{L}_1$-norm over the course of T and must be less than what one compressor can deliver in this interval. Without such a bound on the disturbance the problem of finding controllers becomes infeasible, as the compressors would not be able to supply enough air, regardless of controller.

Remark 3. *Most industrial compressed air networks are secured against critically high pressure levels through blow-off valves. Hence, only the deviation to the lower pressure bound is operation-critical in this case. However, this does not necessarily hold true for different network use cases and remains relevant for energy consumption.*

4.2 Compressors, interconnections and scheduling

Industrial compressor stations consist of several compressors of identical or distinct type and dimension that jointly feed into a pipe network. For different use cases, multiple separated networks can be controlled at different pressure levels. For one network, we define a multi-agent system with N agents $\Sigma_i, i = 1, \ldots, N$ with nonlinear dynamics:

$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i, x_\mathrm{p}, u_i) \\ y_i = h_i(x_i), \end{cases} \quad (21)$$

The systems are interconnected through the joint x_p, which influences and is influenced by each Σ_i. Only active compressors and have variable, i.e., controllable output, are considered. Contributions of compressors with fixed output, e.g., screw compressors, are subtracted from $\tilde{V}_\mathrm{out}$ to arrive at a controllable subsystem. Compressor activation scheduling is abstracted away from (21) by assuming the planned reserve $\sum_{i=1}^{N} y_{i,\max} \ge V_\mathrm{out}$ to be sufficient for the given time frame. We assume that any switching is performed in stable manner. However, due to the high inertia of movable compressor parts, there is a duration T between initiating and completing a switch.

4.3 Controller structure

We are interested in finding distributed controllers which ensure that the distribution system is within a prescribed margin of the pressure. Given N compressors, defined as in (21) we seek to find controllers $k_i(x_i, x_\mathrm{p}, y_{j\neq i})$ such that, with $x = [x_1^\mathrm{T}, \ldots, x_N^\mathrm{T}]^\mathrm{T}$, $y = [y_1^\mathrm{T}, \ldots, y_N^\mathrm{T}]^\mathrm{T}$ and $u = [u_1^\mathrm{T}, \ldots, u_N^\mathrm{T}]^\mathrm{T}$, the network

$$\Sigma : \begin{cases} \dot{x}_i = f_i\left(x_i, x_\mathrm{p}, k_i(x_i, x_\mathrm{p}, y_{j\neq i})\right), \\ \dot{x}_\mathrm{p} = f_\mathrm{p}(h_i(x_i), x_\mathrm{p}, d), \\ y_\mathrm{p} = x_\mathrm{p}, \end{cases} \quad (22)$$

has a bounded $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain from the disturbance d to the output y_p. I.e., we seek controllers k_i such that Σ achieve a specified $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain according to Definition 3, which ensures that the pressure remain within the specified margins. The distributed controller has advantages compared to a single controller with multiple actuators in several ways. First, by calculating the control action locally, large distances between compressors do not require data transmission from a centralized control entity. Second, a distributed controller allows for easier scaling of the plants.

Thus, future research will focus on finding controllers k_i which ensure a prescribed $\mathcal{L}_1 \to \mathcal{L}_\infty$-gain from the disturbance d to the pressure x_p for the system in (22).

5. CONCLUSION

We have presented initial necessary and sufficient condition to ensure a finite gain from $\mathcal{L}_1$ to $\mathcal{L}_\infty$ spaces. We have done this using classical proofs of BIBO-stability and related this to the more widely used $\mathcal{H}_2$-gain. Future research will include ISS-formulation and techniques to find controllers and tighter bounds on this gain.

ACKNOWLEDGEMENTS

We would like to thank Victoria Deichmann and Joelle Christ for many fruitful discussions.

REFERENCES

Ahmed, A., Zagorowska, M., Del Rio-Chanona, E.A., and Mercangoz, M. (2022). Application of gaussian processes to online approximation of compressor maps for load-sharing in a compressor station. In *2022 European Control Conference (ECC)*. IEEE.

Al Zawaideh, A., Al Hosani, K., Boiko, I., and Hammadih, M.L. (2022). Minimum energy adaptive load sharing of parallel operated compressors. *IEEE Open Journal of Industry Applications*, 3.

Al Zawaideh, I.A., Hassan Al Hosani, K., Boiko, I., AlQassab, A., and Khan, I. (2021). Control-oriented modelling and optimal adaptive control of parallel compressors. In *Abu Dhabi International Petroleum Exhibition & Conference 2021*.

Bloch, H.P. (2010). *A practical guide to compressor technology*. Wiley-Interscience, 2nd ed edition.

Briat, C. (2013). Robust stability and stabilization of uncertain linear positive systems via integral linear constraints: l_1-gain and l_∞-gain characterization. *International Journal of Robust and Nonlinear Control*.

Cortinovis, A., Mercangöz, M., Zovadelli, M., Pareschi, D., De Marco, A., and Bittanti, S. (2016). Online performance tracking and load sharing optimization for parallel operation of gas compressors. *Computers & Chemical Engineering*, 88.

J. Luo, e.a. (2022). Controlling commercial cooling systems using reinforcement learning. *arXiv*.

Khalil, H.K. (2002). *Nonlinear Systems*. Prentice Hall, 3 edition.

Kopanos, G.M., Xenos, D.P., Cicciotti, M., Pistikopoulos, E.N., and Thornhill, N.F. (2015). Optimization of a network of compressors in parallel: Operational and maintenance planning – the air separation plant case. *Applied Energy*, 146.

Kôsaku, Y. (1995). *Functional analysis*. Springer-Verlag Berlin Heidelberg New York, 6 edition.

Kousoulidis, D. and Forni, F. (2022). Polyhedral estimation of l-1 and l-infinity incremental gains of nonlinear systems. *arXiv*.

Kumar, S. and Cortinovis, A. (2017). Load sharing optimization for parallel and serial compressor stations. In *2017 IEEE CCTA*.

Li, X., Cui, T., Huang, K., and Ma, X. (2021). Optimization of load sharing for parallel compressors using a novel hybrid intelligent algorithm. *Energy Science & Engineering*, 9(3).

Mahlke, D., Martin, A., and Moritz, S. (2010). A mixed integer approach for time-dependent gas network optimization. *Optimization Methods and Software*, 25(4).

Milosavljevic, P., Marchetti, A.G., Cortinovis, A., Faulwasser, T., Mercangöz, M., and Bonvin, D. (2020). Real-time optimization of load sharing for gas compressors in the presence of uncertainty. *Applied Energy*, 272.

Nazif, S., Karamouz, M., Tabesh, M., and Moridi, A. (2010). Pressure management model for urban water distribution networks. *Water Resources Management*, 24(3).

Nehler, T. (2018). Linking energy efficiency measures in industrial compressed air systems with non-energy benefits – a review. *Renewable and Sustainable Energy Reviews*, 89.

Paparella, F., Dominguez, L., Cortinovis, A., Mercangoz, M., Pareschi, D., and Bittanti, S. (2013). Load sharing optimization of parallel compressors. In *2013 ECC*.

Peyrl, H. and Cortinovis, A. (2015). Computationally efficient solution of a compressor load sharing problem using the alternating direction method of multipliers. In *2015 European Control Conference (ECC)*.

Rodrigues, L.R. (2022). A hybrid multi-population metaheuristic applied to load-sharing optimization of gas compressor stations. *Computers & Electrical Engineering*, 97.

Rotea, M.A. (1993). The generalized h2 control problem. *Automatica*, 29(2).

Saidur, R., Rahim, N., and Hasanuzzaman, M. (2010). A review on compressed-air energy use and energy savings. *Renewable and Sustainable Energy Reviews*, 14(4).

Skogestad, S. and Postlethwaite, I. (2001). *Multi-variable feedback control*. John Wiley & Sons, 2nd ed edition.

Sun, Y., Zhang, B., Ge, L., Sidorov, D., Wang, J., and Xu, Z. (2020). Day-ahead optimization schedule for gas-electric integrated energy system based on second-order cone programming. *CSEE Journal of Power and Energy Systems*, 6(1).

Torrisi, G., Grammatico, S., Cortinovis, A., Mercangoz, M., Morari, M., and Smith, R.S. (2017). Model predictive approaches for active surge control in centrifugal compressors. *IEEE Tran. on Control Systems Technology*, 25(6).

Uddin, M.F., Hassan, K.M.N., and Biswas, S. (2021). Peak load minimization in smart grid by optimal coordinated ON–OFF scheduling of air conditioning compressors. *Sustainable Energy, Grids and Networks*, 28.

Unger, M. and Radgen, P. (2018). Energy efficiency in compressed air systems – a review of energy efficiency potentials, technological development, energy policy actions and future importance. In *Energy Efficiency in Motor Driven Systems (EEMODS) 2017*.

Wong, W., Dutta, P., Voicu, O., Chervonyi, Y., Paduraru, C., and Luo, J. (2022). Optimizing industrial HVAC systems with hierarchical reinforcement learning. *arXiv*.

Xenos, D.P., Cicciotti, M., Kopanos, G.M., Bouaswaig, A.E., Kahrs, O., Martinez-Botas, R., and Thornhill, N.F. (2015). Optimization of a network of compressors in parallel: Real time optimization (RTO) of compressors in chemical plants – an industrial case study. *Applied Energy*, 144.

Zagorowska, M., Degner, M., Ortmann, L., Ahmed, A., Bolognani, S., del Rio Chanona, E., and Mercangöz, M. (2023). Online feedback optimization of compressor stations with model adaptation using gaussian process regression. *Journal of Process Control*, 121.

Zagorowska, M., Skourup, C., and Thornhill, N.F. (2020). Influence of compressor degradation on optimal operation of a compressor station. *Computers & Chemical Engineering*, 143.

Zhang, J., Li, L., Zhang, Q., and Wu, Y. (2022). Optimization of load sharing in compressor station based on improved salp swarm algorithm. *Energies*, 15(15).

Zhao, Y., Xu, X., Qadrdan, M., and Wu, J. (2021). Optimal operation of compressor units in gas networks to provide flexibility to power systems. *Applied Energy*, 290.