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Optimal control in a social interaction model with law enforcement ⋆

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Abstract: In this work, we consider an age-dependent social interaction model and study the stability and optimal control. Law enforcement is an important factor in controlling crime, and the deployment of police to enforce it is important. Now cost is a major issue, so optimal deployment is very important to study. Stability results are derived in terms of threshold parameters. Basic reproduction number is calculated for stability analysis. Using the adjoint system, the form of the law enforcement factor is obtained in terms of state space variables. We observe that the cost functional increases with the increase in the population density of the criminal population. Numerical results are also incorporated to illustrate the theoretical findings.

Keywords: Age-structured model, Optimal control, Law enforcement, Social interaction model.

1. INTRODUCTION

We characterize criminal population according to their age. Age zero means the minimum age at which the criminal population is entering society. Let $C(a,t)$ be the population density of age a individuals at any time t, and $N(t)$ be the number of non-criminal individuals at time t, $L_p(a,t)$ be the law enforcement factor on age a individuals at any time t.

Then the evolution of two populations is modeled by the following initial boundary value problem

$$\frac{\partial C(a,t)}{\partial t} + \frac{\partial C(a,t)}{\partial a} = -\gamma(a)C(a,t) - L_p(a,t)C(a,t)$$

$$\frac{dN(t)}{dt} = \mu N\left(1 - \frac{N}{K}\right) - N\int_0^\infty \zeta(a)C(a,t)da \qquad (1)$$

$$+ \int_0^\infty L_p(a,t)C(a,t)da$$

$$C(0,t) = N\int_0^\infty \zeta(a)C(a,t)da$$

$$N(0) = N_0,\ C(a,0) = C_0(a),$$

where C_0 is the initial population density, $\gamma(a)$ is the natural mortality rate of the criminal population, $\zeta(a)$ is the birth rate of the criminally active population, μ is growth rate of $N(t)$ and K is the carrying capacity.

2. RESULTS

Our focus is to perform stability analysis on two steady states $E^0 = (0,K)$ and $E^* = (C^*(a), N^*)$, where $(C^*(a), N^*)$ is the non trivial solution of system (1). The non-trivial solution exists, because model (1) has existence and uniqueness of solution.

Let

$$R_0 = K\int_0^\infty \zeta(a)e^{-\int_0^a[\gamma(\xi)+L_p(\xi)]d\xi}da.$$

We define R_0 as the threshold parameter. Conditions for the stability of crime free and coexistence steady states are stated in the next two theorems. For proof, we refer to [4], similar methods can be used for the proof.
Theorem1. Crime-free steady state is locally-stable if $R_0 < 1$ and unstable if $R_0 > 1$.
Theorem2. A non-trivial equilibrium solution where the coexistence of both populations is possible is locally-stable if threshold parameter $R_0 > 1$.

For the control problem, L_p is the optimal control variable and we want to derive the form of optimal control variable.

Our task is to minimize criminal population. So, let us define the objective functional in the following manner:

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$$J(L_p) = \int_0^T \int_0^{a_m} \left(A_1 C(a,t) + \frac{A_2}{2} L_p(a,t)^2 \right) dadt, \quad (2)$$

where A_1, A_2 are weight factors and we want to minimize $J(L_p)$. The law enforcement factor L_p lies in the set defined in the following way:

$$\mathcal{L}_P = \{l \in L^\infty((0,a_m) \times (0,T); (0,l_m)),\ a_m, l_m > 0\}. \quad (3)$$

We state the following theorem for the existence of optimal controller, for proof, we refer to [2].

Theorem 3. The optimal control problem (1)-(2) has optimal law enforcement factor L_p^* such that

$$\min_{L_p \in \mathcal{L}_p} J(L_p) = J(L_p^*).$$

Theorem 4. Let L_p^* be an optimal control variable which minimizes the cost functional $J(L_p)$. Then it can be written in the explicit form

$$L_p^* = \max\left(0, \min\left(l_m, \frac{(\lambda_2(t) - \lambda_1(a,t))C(a,t)}{A_2}\right)\right),$$

where λ_1, λ_2 are solutions of adjoint system.

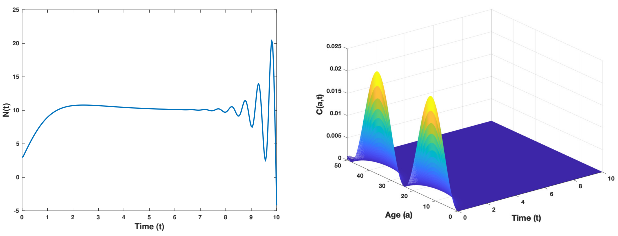
The model is solved using finite difference schemes. An explicit finite difference scheme is used for the purpose of numerical simulations. Initial data taken is of the following form

$$C(a,0) = \alpha(\sin(\beta a))^2 + 0.00001 e^{-a^2}.$$

Birth and mortality rates are taken respectively as

$$\zeta(a) = \frac{0.001a}{(60+a)^2},\ \gamma(a) = 0.005(1 - e^{-5a}).$$

Other parameters taken are $L_p = 3.6, N_0 = 4, K = 10$, and $\mu = 1$.



(a) (b)

Fig. 1. Behaviour of non-criminal population with respect to time. Law enforcement factor taken in Fig. 1(a) is $L_p = 0.8$. In Fig. 1(b), plot of population density of criminally active population is shown.

It is clear that the evolution of the criminal population depends on the initial population distribution and vital rates, and there are two peaks for criminal population. Our theoretical results does not prove the existence of oscillations, but the possible reason could be the high value taken for the law enforcement factor.

3. CONCLUSION

Optimal punishment is important for reducing crime and alleviating the economic burden on a country due to the requirement of a large number of resources. In our work, firstly with the assumption that the law enforcement function is essentially bounded, we show the existence of the solution to the model. It is important to find the conditions on the parameters such that the society attains a crime-free steady state. Although our study is purely mathematical and involves rigorous mathematical techniques, still results are new and interesting. Optimality conditions are derived using adjoint equations and sensitivity equations. The optimal law enforcement factor is obtained in terms of adjoint system and weight factors.

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